Field Theory With a Vector Global Symmetry

Nathan Seiberg

IAS

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Introduction

• Motivated by recent advances with fractons, but not sure this talk is relevant to fractons.
• Study global symmetries, whose conserved charge is a vector.
• One example is the momentum $P^i = \int_{space} T^i_0$. Its conserved Noether current $T^{\mu\nu}$ is symmetric.
• Oneform symmetry is common in relativistic field theories.
• We’ll present a hierarchical set of three symmetries, starting with the most special one and generalizing it.
• For each symmetry (for simplicity $U(1)$) we will present:
  – the conserved currents
  – a class of field theories with the symmetry
  – a concrete example based on a scalar field theory
A lot of earlier work, here we follow [Gaiotto, Kapustin, NS, Willett]

\[ \partial_\mu J^{\mu \nu} = 0 \]

Charges

\[ Q(C) = \int_C J^{\mu \nu} n_{[\mu \nu]} \]

\( C \) a codimension 2 manifold in spacetime orthogonal to \( n_{[\mu \nu]} \). \( Q \) is topological – it does not change under small deformations of \( C \). Specifically, it does not change unless \( C \) crosses another operator.
Relativistic oneform global symmetry

In nonrelativistic terms

\[ \partial_0 J^i_0 - \partial_i J^{[ij]} = 0 \]
\[ G = \partial_i J^i_0 = 0 \]

Charges at fixed time

\[ Q(\mathcal{C}) = \int_{\mathcal{C}} J^j_0 n_j \]

\( \mathcal{C} \) a codimension 1 manifold in space orthogonal to \( n_j \).

\( Q \) is topological – it does not change under small changes of \( \mathcal{C} \).
Relativistic oneform global symmetry

$$\partial_0 J^j_0 - \partial_i J^{[ij]} = 0$$
$$G = \partial_i J^i_0 = 0$$

Example:
Maxwell theory – $U(1)$ gauge theory
$$J^j_0 = F^j_0$$
$$J^{[ij]} = F^{[ij]}$$

$G = 0$ is Gauss law.
$Q(\mathcal{C})$ is the electric flux through $\mathcal{C}$.
The charged operators are Wilson lines.
There is also a magnetic symmetry, but we will not discuss it here.
Relativistic oneform global symmetry

Example: $U(1)$ lattice gauge theory

$p, l, s$ are plaquettes, links, and sites

In $A_0 = 0$ the Hamiltonian is

$$\mathcal{H} = \sum_p (U_p + U_p^+) + \sum_l E_l^2$$

The variables are $U(1)$ elements $U_l$ and their conjugate momenta $E_l$. $U_p = \prod_{l \subset p} U_l$ (oriented product)

Impose Gauss law  

$$G = \sum_{l \supset s} E_l = 0 \text{ (oriented sum)}$$

$Q(\mathcal{C}) = \sum_{l \subset \mathcal{C}} E_l$ (oriented sum over the links pierced by $\mathcal{C}$) is the electric flux through $\mathcal{C}$. It is conserved and topological.
As before, 

\[ \partial_0 J^j_0 - \partial_i J^{[ij]} = 0 , \]

but now we do not impose 

\[ G = \partial_i J^i_0 = 0 . \]

The conservation leads to 

\[ \partial_0 G = \partial_i \partial_0 J^i_0 = 0 , \]

e.g. \( G \) is conserved at every point, but it is not zero.

As before, conserved charges at fixed time 

\[ Q(C) = \int_C J^j_0 n_j , \]

but now \( Q \) is not topological.

A cruder charge is 

\[ Q^j = \int_{space} J^j_0 . \]

Correspondingly, point operators can transform under \( Q(C), Q^j \).
Nonrelativistic oneform global symmetry

\[ \partial_0 J_0^j - \partial_i J^{[ij]} = 0 \]

Therefore

\[ \partial_0 G = \partial_i \partial_0 J_0^i = 0 \]

\[ Q(\mathcal{C}) = \int_{\mathcal{C}} J_0^j n_j \]

If there is an \( \mathcal{M} \), such that \( \mathcal{C} = \partial \mathcal{M} \)

\[ Q(\mathcal{C}) = \int_{\mathcal{M}} \partial_i J_0^i = \int_{\mathcal{M}} G \]

The conservation of \( Q(\mathcal{C}) \) implies the conservation of \( G \).

But the conservation of \( Q(\mathcal{C}) \) contains more information, because \( \mathcal{C} \) might not be a boundary.
Nonrelativistic oneform global symmetry

Lattice example:

$U(1)$ lattice gauge theory in $A_0 = 0$, but without imposing Gauss law $G = \sum_{l \supseteq s} E_l = 0$ [Kitaev]

$$\mathcal{H} = \sum_p \left( U_p + U_p^+ \right) + \sum_l E_l^2 + \sum_s \left( \sum_{l \supseteq s} E_l \right)^2$$

The last term imposes Gauss law energetically.

Interpret as $U(1)$ gauge theory with charged matter at the lattice scale.

$G = \sum_{l \supseteq s} E_l$, is conserved, but it is nonzero.

$Q(C) = \sum_{l \subseteq C} E_l$ are conserved, but they are not topological.

Related discussion in [Hermele, Fisher, Balents; Williamson, Bi, Cheng].
Nonrelativistic oneform global symmetry
A class of continuum examples

\[ \partial_0 J_0^j - \partial_i J_{[ij]} = 0 \]

Couple the \( U(1) \) gauge theory to charged matter fields, such that we still have

\[
\begin{align*}
J_0^j &= F_0^j \\
J_{[ij]} &= F_{[ij]}
\end{align*}
\]

For that, the matter theory should have an operator \( \mathcal{O} \) satisfying \( \partial_0 \mathcal{O} = 0 \).

The \( U(1) \) gauge theory couples to \( (J_0^{\text{matter}} = \mathcal{O}, J^i_{\text{matter}} = 0) \)

\[
\mathcal{L}_1 = \mathcal{L}_0 + (F_0^i)^2 - (F^{ij})^2 + A_0 \mathcal{O}
\]

with \( \mathcal{L}_0 \) the matter Lagrangian. This is \( U(1) \) gauge invariant and has the nonrelativistic oneform symmetry with \( G = \partial_i J_0^i = \mathcal{O} \).
Nonrelativistic oneform global symmetry

A class of continuum examples

\[ \partial_0 \mathcal{O} = 0 \]

Such a matter theory has infinitely many conserved charges. \( C(x^i) \) is conserved for every \( C(x^i) \).

The charged matter is not mobile.

A concrete example:

A complex scalar field \( \Phi \) with

\[ \mathcal{L}_0 = i\bar{\Phi} \partial_0 \Phi - \partial_i (\bar{\Phi} \Phi) \partial^i (\bar{\Phi} \Phi) - |\Phi|^4 + \ldots \]

No \( \partial_i \bar{\Phi} \partial^i \Phi \) term. Highly nonstandard. Similar to fractons.

Invariance under \( \Phi \rightarrow e^{iC(x^i)} \Phi \)

The charge density \( \mathcal{O} = |\Phi|^2 \) at a point is conserved.
A more general vector symmetry

Previous case \[ \partial_0 J^j_0 - \partial_i J^{ij} = 0 \]
with antisymmetric \( J^{ij} \).

Generalize to \( J^{ij} \) with no restriction on the symmetry of \( ij \).

Now \[ \partial_0 G = \partial_i \partial_0 J^i_0 = \partial_i \partial_j J^{ij} \neq 0 . \]

Therefore, the charge operators on codimension one manifold \( \mathcal{C} \)
\[ Q(\mathcal{C}) = \int_{\mathcal{C}} J^j_0 n_j \] are no longer conserved, but the cruder charge
\[ Q^j = \int_{space} J^j_0 \]
is conserved.
A more general vector symmetry

A class of continuum examples

\[ \partial_0 J^0_j - \partial_i J^{ij} = 0 \]

For \( J^{(ij)} = 0 \), we coupled charged matter with \( \partial_0 \mathcal{O} = 0 \) to a \( U(1) \) gauge field.

Now, we take a matter theory with

\[ \partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0 \quad \text{(with } \mathcal{O}^{(ij)}) \]

Couple the \( U(1) \) gauge field to \( (J_0^{\text{matter}} = \mathcal{O}, J^i_{\text{matter}} = \partial_j \mathcal{O}^{ij}) \)

\[ \mathcal{L}_1 = \mathcal{L}_0 + (F^i_0)^2 - (F^{ij})^2 + A_0 \mathcal{O} - A_i \partial_j \mathcal{O}^{ij} \]

The conserved current of the global symmetry are

\[ J^j_0 = F^j_0 \]

\[ J^{ij} = F^{[ij]} - \mathcal{O}^{ij} \]
A more general vector symmetry

A class of continuum examples

\[ \partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0 \]

Such a matter theory (before coupling to the gauge field) has the conserved charges

\[ Q = \int_{\text{space}} J_0^{\text{matter}} = \int_{\text{space}} \mathcal{O} \]

\[ Q^j = \int_{\text{space}} x^j J_0^{\text{matter}} = \int_{\text{space}} x^j \mathcal{O} \]

They can be interpreted as:

• a global $U(1)$ symmetry (which we gauge)
• a vector symmetry, dipole symmetry, with charge $Q^j$

After the gauging we are left only with the vector symmetry.
A more general vector symmetry
A class of continuum examples

$$\partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0$$

This defining equation is local and therefore, the discussion makes sense on every manifold.

This is not true for the charge, $Q^j = \int_{space} x^j \mathcal{O}$, which makes sense only on $\mathbb{R}^D$.

After coupling to the $U(1)$ gauge field the conserved current of the vector symmetry is

$$J_0^j = F_0^j$$
$$J[ij] = F[ij] - \mathcal{O}^{ij}$$

No explicit $x^j$ dependence.
A more general vector symmetry

A concrete continuum example [Pretko]

\[ \partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0 \]

A complex scalar field \( \Phi \) with

\[ \mathcal{L}_0 = i \Phi \partial_0 \Phi - \partial_i (\Phi \Phi) \partial^i (\Phi \Phi) - |\Phi|^4 \\
+ i(\Phi^2 \partial_i \Phi \partial^i \Phi - \Phi^2 \partial_i \Phi \partial^i \Phi) + \ldots \]

Again, no \( \partial_i \Phi \partial^i \Phi \) term.

The new term \( i(\Phi^2 \partial_i \Phi \partial^i \Phi - \Phi^2 \partial_i \Phi \partial^i \Phi) \) breaks the symmetry \( \Phi \to e^{iC(x^i)} \Phi \) to \( \Phi \to e^{i\alpha + ic_i x^i} \Phi \).

Here

\[ \mathcal{O} = |\Phi|^2 \]
\[ \mathcal{O}^{ij} = |\Phi|^4 \delta^{ij} + \ldots \]

Higher order terms can lead to a traceless symmetric tensor in \( \mathcal{O}^{ij} \).
Summary of the symmetries

$$\partial_0 J_0^j - \partial_i J_{ij}^i = 0$$

- A vector symmetry: $J_{ij}^i$ not restricted
  
  The charge
  $$Q^j = \int_{\text{space}} J_0^j$$

- Nonrelativistic oneform symmetry: impose also
  $$j_{ij}^i = -J_{ji}^j$$

  $Q(C) = \int_C J_0^j n_j$ is associated with a nontopological manifold $C$

- As in the relativistic symmetry: impose also
  $$\partial_j J_0^j = 0$$

  $Q(C) = \int_C J_0^j n_j$ is associated with a topological manifold $C$
Gauging

Start with a theory with \( \partial_0 J^j_0 - \partial_i J^{ij} = 0 \).

We gauge by introducing sources \( B_0_j, A_{ij} = \frac{1}{2} (B_{[ij]} + S_{(ij)}) \).

(No need to introduce \( S_{(ij)} \) when \( J^{(ij)} = 0 \).

Add to the Lagrangian the minimal coupling terms \( B_0_j J^j_0 - A_{ij} J^{ij} \)

Invariance under the gauge symmetry
\[
B_0_i \rightarrow B_0_i + \partial_0 c_i \\
A_{ij} \rightarrow A_{ij} + \partial_i c_j
\]

• No \( c_0 \)
• \( A_{ij} \) has no symmetry in \( ij \) – peculiar gauge field
• \( c_i \) is not a \( U(1) \) gauge field
Gauging

\[ B_{0i} \rightarrow B_{0i} + \partial_0 c_i \]
\[ A_{ij} \rightarrow A_{ij} + \partial_i c_j \]

We can also add kinetic terms for these fields using the gauge invariant field strengths

\[ \epsilon_{ij} = \partial_0 A_{ij} - \partial_i B_{0j} \]
\[ \beta_{ijk} = \partial_i A_{jk} - \partial_j A_{ik} \]

The standard \( H_{\mu \nu \rho} \) are linear combinations of them

\[ H_{0ij} = \epsilon_{ij} - \epsilon_{ji} \]
\[ H_{ijk} = \beta_{ijk} - \beta_{ikj} + \beta_{jki} \]
Let us implement it explicitly in the $U(1)$ theory coupled to a matter system. We started with a matter theory with
\[ \partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0 \]
Above we gauged an ordinary $U(1)$ with
\[
J^0_{\text{matter}} = \mathcal{O}, \quad J^i_{\text{matter}} = \partial_j \mathcal{O}^{ij},
\]
i.e.
\[
\mathcal{L}_1 = \mathcal{L}_0 + (F_0^i)^2 - (F^{ij})^2 + A_0 \mathcal{O} - A_i \partial_j \mathcal{O}^{ij}
\]
The remaining global symmetry
\[
J_0^j = F_0^j
\]
\[
J^{[ij]} = F^{[ij]} - \mathcal{O}^{ij}
\]
Now, gauge this remaining vector global symmetry.
Gauging

\[ J_0^j = F_0^j \]
\[ J^{ij} = F^{[ij]} - \mathcal{O}^{(ij)} \]

Add new gauge fields \( B_{0j} \), \( A_{ij} = \frac{1}{2} (B_{[ij]} + S_{(ij)}) \) with minimal coupling

\[ \mathcal{L}_2 = \mathcal{L}_0 + (F_0^i + B_0^i)^2 - (F^{ij} + B^{ij})^2 + A_0\mathcal{O} + A_{ij}\mathcal{O}^{ij} + \ldots \]

where \( A_{ij} = \frac{1}{2} (S_{ij} + \partial_i A_j + \partial_j A_i) \).

The \( U(1) \) gauge symmetry acts as

\[ A_0 \rightarrow A_0 + \partial_0 \alpha \]
\[ A_i \rightarrow A_i + \partial_i \alpha \]
\[ A_{ij} \rightarrow A_{ij} + \partial_i \partial_j \alpha \]

In addition, we also have the gauge symmetry...
Gauging

Additional gauge symmetry

\[ A_i \rightarrow A_i + c_i \]
\[ B_{0i} \rightarrow B_{0i} + \partial_0 c_i \]
\[ B_{ij} \rightarrow B_{ij} + \partial_i c_j - \partial_j c_i \]

Note, \( A_{ij} = \frac{1}{2} \left( S_{ij} + \partial_i A_j + \partial_j A_i \right) \) is invariant under \( c_i \)

Can add kinetic terms using

\[ H_{0ij} = \partial_0 B_{ij} - \partial_i B_{0j} + \partial_j B_{0i} \]
\[ H_{ijk} = \partial_i B_{jk} - \partial_j B_{ik} + \partial_k B_{ij} \]
\[ \mathcal{E}_{ij} = \partial_0 A_{ij} - \partial_i \partial_j A_0 \]
\[ \mathcal{B}_{ijk} = \partial_i A_{jk} - \partial_j A_{ik} \]

\( A_i \) is “Higgsed” and is lifted (becomes massive) with \( B_{i0} \) and \( B_{ij} \).

We are left with \( A_0 \) and \( A_{ij} \).
Gauging

We can do the gauging in one step. (Similar to discussions in [Rasmussen, You, Xu; Pretko; Slagle, Prem, Pretko; ...].)

We started with a matter theory with
\[ \partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0 \]

We couple these operators to sources
\[ \mathcal{L}_0 + A_0 \mathcal{O} + A_{ij} \mathcal{O}^{ij} + \cdots \]

Hence, there is a $U(1)$ gauge symmetry
\[ A_0 \rightarrow A_0 + \partial_0 \alpha \]
\[ A_{ij} \rightarrow A_{ij} + \partial_i \partial_j \alpha \]

And we can add kinetic terms using
\[ \mathcal{E}_{ij} = \partial_0 A_{ij} - \partial_i \partial_j A_0 \]
\[ \mathcal{B}_{ijk} = \partial_i A_{jk} - \partial_j A_{ik} \]
Summary

• A hierarchy of global symmetries, whose charges $Q^i$ carry a spatial vector index.
• For every one of these we presented a large class of theories exhibiting them.
  – All these examples are based on a $U(1)$ gauge theory coupled to a special matter theory
  – We showed concrete examples of these matter theories
• We can gauge these new symmetries. The needed gauge field is an antisymmetric tensor $B_{[\mu\nu]}$ and in the general symmetry we also need a symmetric tensor $S_{(ij)}$. 