Confinement, De-confinement, and 3d Topological Quantum Field Theory

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IAS
\[ \frac{1}{4g^2} \int Tr (F \wedge* F) + \frac{i \theta}{8\pi^2} \int Tr(F \wedge F) \]

- \( \theta \) is 2\( \pi \)-periodic (on a closed manifold)
- Generic \( \theta \)
  - Unique vacuum (no TQFT at low energies), gapped spectrum
  - Confinement: \( \langle W \rangle = \langle Tr e^{i\oint a} \rangle \sim e^{-Area} \rightarrow 0 \) at long distances
- \( \theta \) multiple of \( \pi \): CP symmetry (equivalently, time-reversal)
- \( \theta \) odd multiple of \( \pi \)
  - CP is spontaneously broken – 2 vacua
  - Domain walls
$SU(N)$ pure gauge theory in $4d$

Additional observables

[Kapustin, NS; Gaiotto, Kapustin, NS, Willett]

- Study $PSU(N)$ bundles that are not $SU(N)$ bundles – nontrivial $w_2$ of the bundle.
- Can describe using a $\mathbb{Z}_N$ classical background (two-form) gauge field $B$ that sets $w_2 = B$.
- $B$ can be interpreted as a classical background gauge field of a $\mathbb{Z}_N$ one-form global symmetry. More below.
- This is not a $PSU(N)$ gauge theory, where $B$ is summed over. More below.
\( SU(N) \) pure gauge theory in 4\( d \)

The operators [...; Kapustin, NS]

- Wilson lines \( W(C) = \text{Tr}(e^{i \phi_C} a) e^{i \int_\Sigma B} \) with \( C = \partial \Sigma \)
- Charges: topological surface operators \( U_E(X) = e^{i \phi_X u(a)} \)
- The ‘t Hooft operator \( T \) is not a genuine line operator.
  - Since a Wilson line can detect the Dirac string emanating from the ‘t Hooft operator, the Dirac string is visible and sweeps a surface \( \Sigma \)
    \[
    T(C) e^{i \int_\Sigma u(a)}
    \]
    It is an open version of \( U_E \).
- \( U_E \) can be interpreted as the worldsheet of a Dirac string.
\( SU(N) \) pure gauge theory in 4\( d \)

\( B \)-dependent counterterm

[Kapustin, NS; Gaiotto, Kapustin, NS, Willett; Gaiotto, Kapustin, Komargodski, NS]

• Can add to the action a counterterm in the classical fields:

\[
\frac{2\pi i}{2N} \int \mathcal{P}(B)
\]

\( p = 1, 2, \ldots, 2N, \ pN \in 2\mathbb{Z} \) (\( \mathcal{P} \) is the Pontryagin square)

• For nonzero \( B \) there are fractional instantons and therefore \( \theta \) is not 2\( \pi \)-periodic. Instead, \( (\theta, p) \sim (\theta + 2\pi, p + N - 1) \)

• \( \theta \rightarrow \theta + 2\pi \) leads to different theories
  – Different contact terms
  – Different behavior on boundaries
SU(N) pure gauge theory in 4d

CP at \( \theta = \pi \) [Gaiotto, Kapustin, Komargodski, NS]

\[
\frac{i \theta}{8\pi^2} \int \text{Tr}(F \wedge F) + \frac{2\pi i p}{2N} \int \mathcal{P}(B) \\
(\theta, p) \sim (\theta + 2\pi, p + N - 1)
\]

\( \mathcal{CP}: (\pi, p) \to (-\pi, -p) \sim (\pi, -p + N - 1) \)

• For \( N \) even, no value of \( p \) preserves the symmetry – mixed anomaly between \( \mathcal{CP} \) and the \( \mathbb{Z}_N \) one-form symmetry
• \( CP \) is spontaneously broken with 2 vacua
• Nontrivial domain wall between the two vacua at \( \theta = \pi \)
  – Anomaly inflow from the bulk
  – \( SU(N)_1 \) Chern-Simons theory on the wall.
More generally, consider a space-dependent \( \theta \) interpolating between \( \theta = 0 \) and \( \theta = 2\pi k \) for some integer \( k \):

- If the interpolation is very slow (compared with the confinement scale of the theory), we have \( k \) copies of \( SU(N)_1 \) localized where \( \theta \) crosses an odd multiple of \( \pi \).
If the interpolation is fast compared with the confinement scale of the theory, we can have another TQFT $\mathcal{T}_k$.

Need better dynamical control to determine $\mathcal{T}_k$. One option is $\mathcal{T}_k = SU(N)_k$. This is consistent with the anomaly flow from the bulk.

If the interface is sharp (discontinuous), the result is not universal, but it must have the same anomaly.
SU(N) pure gauge theory in 4d

Interface

\[ \theta = 0 \]

\[ \mathcal{T}_k = \{ SU(N)_k \? \} \]

- Can think of it also as separating regions with the same \( \theta \), but different values of \( p \).
On one side of the interface monopoles condense and lead to confinement. On the other side dyons condense and lead to “oblique confinement.”

By continuity, none of them condense on the interface and hence no confinement there.

- The $SU(N)_k$ Wilson lines are Wilson lines of the microscopic gauge theory.
- Surprise: they have nontrivial braiding – probe quarks are anyons.
Consider a $\mathbb{Z}_N$ charge operator $SSE X = e^{i \oint X u_a}$ that pierces the interface.

We interpreted it as the worldsheet of a Dirac string.

It is associated with a monopole on one side and a dyon on the other side. Therefore, it has electric charge $\pi$ on the wall. It is a Wilson line there.

This explains why there is braiding between Wilson lines on the wall.
**PSU(N)** pure gauge theory in 4d

Gauge the \( \mathbb{Z}_N \) one-form global symmetry

[Kapustin, NS; Gaiotto, Kapustin, NS, Willett]

- Make \( B \) dynamical and denote it by \( b \). This amounts to summing over \( w_2 \) of the \( PSU(N) \) bundles
- The Wilson line is not a genuine line operator
  \[
  W(C, \Sigma) = \text{Tr}(e^{i \oint_C a}) e^{i \int_\Sigma b}
  \]
  with \( C = \partial \Sigma \)
- The correlation functions of \( U_E(X) = e^{i \oint_X u(a)} \) are trivial.
- For \( p = 0 \) the ‘t Hooft operator is a genuine line operator
  \[
  T(C) e^{i \int_\Sigma u(a)}
  \]
  because there is no dependence on \( \Sigma \). (For other values of \( p \) it should be multiplied by a Wilson line.)
**PSU(N) pure gauge theory in 4d**

Gauge the $\mathbb{Z}_N$ one-form global symmetry

- New surface operator
  
  \[ U_M = e^{i \oint_X b} = e^{i \oint_X w_2} \]

- It generates a magnetic $\mathbb{Z}_N$ one-form symmetry

- \( \langle U_M(X) T(C) \rangle = e^{\frac{2\pi i}{N} \langle X, C \rangle} \langle T(C) \rangle \)

\( \langle X, C \rangle \) the linking number.

- The Wilson line \( W(C, \Sigma) = \text{Tr}(e^{i \oint_C a}) e^{i \int_{\Sigma} b} \) is an open version of \( U_M \)
\( PSU(N) \) pure gauge theory in \( 4d \) Dynamics

[Aharony, Tachikawa, NS; Kapustin, NS; Gaiotto, Kapustin, NS, Willett]

\[
\frac{i \theta}{8\pi^2} \int \text{Tr}(F \wedge F) + \frac{2\pi i \, p}{2N} \int \mathcal{P}(b) \\
(\theta, p) \sim (\theta + 2\pi, p + N - 1)
\]

Now the lack of \( 2\pi \)-periodicity in \( \theta \) is more important.

- For \( p = 0 \) monopoles condense, the ‘t Hooft operator has a perimeter law
  - Gapped spectrum, but a nontrivial TQFT at low energies (not merely an SPT phase) – a \( \mathbb{Z}_N \) gauge theory
- More generally, the low energy theory is a \( \mathbb{Z}_L \) gauge theory (could be twisted on nonspin manifolds) with \( L = \gcd(p, N) \)
In the $SU(N)$ theory

\[ \theta = 0 \]
\[ p^{-} \]

\[ \mathcal{T}_k \ (= SU(N)_k?) \]

In the $PSU(N)$ theory

\[ \theta = 0 \]
\[ p^{-} \]

$\mathbb{Z}_{L^-}$ gauge theory

\[ L^\pm = \gcd(p^\pm, N) \]

Cannot have $\frac{\mathcal{T}_k}{\mathbb{Z}_N} \ (= PSU(N)_k?)$ on the interface – it is not consistent!
In order to figure out the theory on the interface we need to understand better the one-form global symmetry, its anomaly, and its gauging.
A 3d TQFT $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form global symmetry [Gaiotto, Kapustin, NS, Willett]

There are line operators $U_l$ with $l$ integer modulo $N$ such that

- $U_l U_{l'} = U_{l+l'}$
- Any line $W \in \mathcal{T}$ has a $\mathbb{Z}_N$ charge $q(W)$ ($\mathbb{Z}_N$ representation)

\[ = e^{\frac{2\pi i q(W) l}{N}} \]
A 3d TQFT $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form global symmetry [Gaiotto, Kapustin, NS, Willett]

Examples

- $SU(N)_k$ has a $\mathbb{Z}_N$ one-form symmetry associated with the center of the gauge group. It is generated by a line in a representation of $k$ symmetric fundamentals.

- $U(1)_N$ has $N$ lines (for $N$ even) realizing a $\mathbb{Z}_N$ one-form symmetry, generated by the line of charge one.

- $\mathbb{Z}_N$ gauge theory has a $\mathbb{Z}_N \otimes \mathbb{Z}_N$ one-form symmetry (electric and magnetic)
A 3d TQFT $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form global symmetry [Gomis, Komargodski, NS; Hsin, Lam, NS]

Consistency implies that the spins of the charge lines are

$$h(U_l) = \frac{p \, l^2}{2N} \mod 1$$

Here $p = 0, 1, \ldots, 2N, pN \in 2\mathbb{Z}$.

(In a spin TQFT ignore the second condition and $p \sim p + N$. By changing the $\mathbb{Z}_N$ generator we can relate different values of $p$.)

Using the braiding we interpret

$$p \mod N = q(U_1)$$

as the $\mathbb{Z}_N$ charge of the generator of the $\mathbb{Z}_N$ symmetry.

If $p \neq 0$, we cannot gauge the one-form symmetry. $p$ characterizes the ‘t Hooft anomaly of the symmetry.
A 3d TQFT $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form global symmetry

$h(U_l) = \frac{p l^2}{2N} \mod 1$

$p$ characterizes the ‘t Hooft anomaly of the symmetry. Coupling to a background $\mathbb{Z}_N$ gauge field $B$ the corresponding anomaly is our 4d bulk term

$$\frac{2\pi i p}{2N} \int \mathcal{P}(B)$$
A 3d TQFT $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form global symmetry with $p = 0$

$$h(U_l) = \frac{p \ l^2}{2N} \mod 1$$

For $p = 0$

• the lines $U_l$ are $\mathbb{Z}_N$ neutral
• their spins vanish modulo an integer
• their braiding is trivial
• there is no ‘t Hooft anomaly
• We can gauge the symmetry
Gauge the $\mathbb{Z}_N$ one-form global symmetry when $p = 0$ [Moore, NS]

For $p = 0$ we can gauge the symmetry (known in the condensed matter literature as “anyon condensation”)

- Remove from $\mathcal{T}$ all the $\mathbb{Z}_N$ charged lines ($q(W) \neq 0 \mod N$)
- Identify the lines $W \sim U_1 W$
- If a line $W$ is the same as $U_1 W$, it appears multiple times

For our problem with the interface in $PSU(N)$ we need to gauge a TQFT with nonzero $p$. 
A 3d TQFT $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form global symmetry with $p \neq 0$

$$h(U_l) = \frac{p \ l^2}{2N} \mod 1$$

For $p = N$ we can use essentially the standard gauging (except the identification) to find a spin TQFT.

For $L = \gcd(p, N) \neq N$ the lines $l = \frac{N}{L} \hat{l}$ lead to a $\mathbb{Z}_L \subset \mathbb{Z}_N$ subgroup, whose $p$ is $\hat{p} = \frac{pN}{L} = 0 \mod L$.

It can be gauged as above.

The resulting theory has $\mathbb{Z}_{N'}$ one-form symmetry with anomaly $p'$ with $N' = N/L$, $p' = p/L$ and hence $L' = \gcd(N', p') = 1$.

So how should we deal with $L = \gcd(p, N) = 1$?
A 3d TQFT $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form global symmetry with $L = \gcd(p, N) = 1$ \cite{Hsin, Lam, NS}

$$h(U_l) = \frac{p \cdot l^2}{2N} \mod 1$$

For every $W \in \mathcal{T}$ with charge $q(W)$ the line $U_l^r W = W'$ (need to show that $W'$ is unique) with $rp + q(W) = 0 \mod N$ is $\mathbb{Z}_N$ neutral.

- Hence, $\mathcal{T} = \mathcal{T}' \otimes \mathcal{A}^{N,p}$
  - $\mathcal{T}'$ includes all the neutral lines $W'$
  - $\mathcal{A}^{N,p}$ is a minimal TQFT with $\mathbb{Z}_N$ symmetry with anomaly $p$.
  - The factorization is also guaranteed by a theorem of [Muger; Drinfeld, Gelaki, Nikshych, Ostrik]

- This is quite surprising. All the information about the symmetry is in a decoupled universal sector $\mathcal{A}^{N,p}$!
The minimal 3d TQFT with a $\mathbb{Z}_N$ one-form global symmetry with anomaly $p$ with $\text{gcd}(p, N) = 1$

$\mathcal{A}^{N,p}$ [Moore, NS; Hsin, Lam, NS]

Examples

- $U(1)_N = \mathcal{A}^{N,1}$
- $SU(N)_1 = \mathcal{A}^{N,N-1}$
- $U(1)_{Np} = \mathcal{A}^{N,p} \otimes \mathcal{A}^{p,N}$ for $\text{gcd}(p, N) = 1$ (this generalizes $\mathbb{Z}_{Np} = \mathbb{Z}_N \otimes \mathbb{Z}_p$)

$\mathcal{A}^{N,p} \otimes \mathcal{A}^{N,-p}$ has $N^2$ lines with an anomaly free diagonal $\mathbb{Z}_N$ one-form symmetry.

Gauging it leads to $(\mathcal{A}^{N,p} \otimes \mathcal{A}^{N,-p})/\mathbb{Z}_N$, which is a trivial theory.
Using $\mathcal{A}^{N,p}$ [Hsin, Lam, NS]

Starting with a theory $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form symmetry with anomaly $p$ such that $\gcd(p, N) = 1$, we cannot gauge the symmetry.

However, since

$$\mathcal{T} = \mathcal{T}' \otimes \mathcal{A}^{N,p}$$

we can extract $\mathcal{T}'$ either by dropping $\mathcal{A}^{N,p}$, or by tensoring another factor and then gauging an anomaly free $\mathbb{Z}_N$ symmetry

$$\mathcal{T}' = (\mathcal{T} \otimes \mathcal{A}^{N,-p})/\mathbb{Z}_N$$

Since $\mathcal{A}^{N,p}$ is minimal, this procedure is canonical.
Back to the interface in $4d$ $PSU(N)$

For simplicity, let $L^\pm = 1$.
Then the bulk theories are trivial and there must be a $3d$ TQFT on the interface.

(The analysis for generic $L^\pm$ is more subtle and is explained in the paper.)
Back to the interface in $4d$ $PSU(N)$

For simplicity, let $L^\pm = \gcd(p^\pm, N) = 1$.

\[ \theta = 0 \]
\[ p^- \]

\[ \theta = 0 \]
\[ p^+ = p^- - k(N - 1) \]

Since $p^+ - p^- = -k(N - 1) = -p(\mathbb{Z}_N \subset \mathcal{T}_k)$, the diagonal $\mathbb{Z}_N$ in the numerator is anomaly free and can be gauged.

Can interpret $\mathcal{A}^{N,-p^-} \otimes \mathcal{A}^{N,p^+}$ as arising from the bulk on the left and the right such that we can perform the gauging.
Conclusions

4d $SU(N)$ gauge theory

• For generic $\theta$ the spectrum is gapped with a trivial low–energy theory and at $\theta = \pi$ there are two vacua.

• $\mathbb{Z}_N$ one form global symmetry
  – It is unbroken (the theory is confining)
  – We can couple the theory to a background two-form $\mathbb{Z}_N$ gauge field $B$ and add a counterterm $\frac{2\pi i p}{2N} \int \mathcal{P}(B)$
  – Keeping track of this term, $\theta$ is $2\pi N$-periodic ($4\pi N$-periodic for even $N$ on a non-spin manifold).

• Steep interface from $\theta = 0$ to $\theta = 2\pi k$ has a TQFT (e.g. $SU(N)_k$) on it
Conclusions

4d $PSU(N)$ gauge theory is obtained by gauging the $\mathbb{Z}_N$ one-form global symmetry of the $SU(N)$ theory.

- The low energy theory is a $\mathbb{Z}_L$ gauge theory with $L = \gcd(p, N)$
- Interfaces have more subtle TQFTs on them
Conclusions

A 3d TQFT with a $\mathbb{Z}_N$ one-form global symmetry

- It is characterized by an integer $p \mod 2N$, which determines
  - The charge of the generating line
  - The spins of the charge lines
  - The ‘t Hooft anomaly

- For $\gcd(p, N) = 1$ there is a minimal TQFT with $\mathbb{Z}_N$ one-form symmetry and anomaly $p$, $\mathcal{A}^{N,p}$
  - Any theory $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form symmetry with anomaly $p$, such that $\gcd(p, N) = 1$, factorizes $\mathcal{T} = \mathcal{T}' \otimes \mathcal{A}^{N,p}$