D-Branes on Cones and
Gauge/String Dualities

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QCD and String Theory

- At short distances, must smaller than 1 fermi, the quark-antiquark potential is approximately Coulombic, due to the Asymptotic Freedom.
- At large distances the potential should be linear (Wilson) due to formation of confining flux tubes.
Flux Tubes in QCD

- These objects may be approximately described by the Nambu strings (animation from lattice work by D. Leinweber et al, Univ. of Adelaide)

- The tubes are widely used, for example, in jet hadronization algorithms (the Lund String Model) where they snap through quark-antiquark creation.
Large N Gauge Theories

- Connection of gauge theory with string theory is strengthened in 't Hooft’s generalization from 3 colors (SU(3) gauge group) to N colors (SU(N) gauge group).
- Make N large, while keeping the 't Hooft coupling fixed.
- The probability of snapping a flux tube by quark-antiquark creation (meson decay) is $1/N$. The string coupling is $1/N$.
- Yet, the planar diagrams needed in the large N limit are very difficult to sum explicitly.
D-Branes vs. Geometry

- Dirichlet branes (Polchinski) led string theory back to gauge theory in the mid-90’s.
- A stack of $N$ Dirichlet 3-branes realizes $\mathcal{N}=4$ supersymmetric SU($N$) gauge theory in 4 dimensions. It also creates a curved background of 10-d theory of closed superstrings (artwork by E.Imeroni)

\[
 ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \left(-(dx^0)^2 + (dx^i)^2\right) + \left(1 + \frac{L^4}{r^4}\right)^{1/2} \left(dr^2 + r^2 d\Omega_5^2\right)
\]

which for small $r$ approaches $AdS_5 \times S^5$

- Successful matching of graviton absorption by D3-branes, related to 2-point function of stress-energy tensor in the SYM theory, with a gravity calculation in the 3-brane metric (IK; Gubser, IK, Tseytlin) was a precursor of the AdS/CFT correspondence.
Conformal Invariance

- In the $\mathcal{N}=4$ SU(N) SYM theory there are 3 adjoint chiral superfields $Z^i$ coupled to the $\mathcal{N}=1$ SU(N) SYM theory with superpotential $\text{Tr } Z^1 [Z^2, Z^3]$.

- The Asymptotic Freedom is canceled by the extra fields; the beta function is exactly zero! Hence, the theory is invariant under scale transformations $x^\mu \rightarrow \lambda x^\mu$. It is also invariant under space-time inversions.

- Such a theory is called a Conformal Field Theory (CFT).

- The $\mathcal{N}=4$ SU(N) SYM is also invariant under the SU(4) R-symmetry. Its full super-conformal symmetry is SU(2,2|4).
The AdS/CFT duality
Maldacena; Gubser, I.K., Polyakov; Witten

- Relates conformal gauge theory in 4 dimensions to string theory on 5-d Anti-de Sitter space times a 5-d compact space. For the $\mathcal{N}=4$ SYM theory this compact space is a 5-sphere realizing the SU(4) R-symmetry.

- The SO(2,4) geometrical symmetry of the AdS$_5$ space realizes the conformal symmetry of the gauge theory.

- The d-dimensional AdS space is a hyperboloid

\[(X^0)^2 + (X^d)^2 - \sum_{i=1}^{d-1} (X^i)^2 = L^2.\]

- Its metric is

\[ds^2 = \frac{L^2}{z^2} \left( dz^2 - (dx^0)^2 + \sum_{i=1}^{d-2} (dx^i)^2 \right)\]
• When a gauge theory is strongly coupled, the radius of curvature of the dual AdS$_5$ and of the 5-d compact space becomes large:

\[
\frac{L^2}{\alpha'} \sim \sqrt{g_{YM}^2 N}
\]

• String theory in such a weakly curved background can be studied in the effective (super)-gravity approximation, which allows for a host of explicit calculations. Corrections to it proceed in powers of

\[
\frac{\alpha'}{L^2} \sim \lambda^{-1/2}
\]

• Feynman graphs instead develop a weak coupling expansion in powers of $\lambda$. At weak coupling the dual string theory becomes difficult.
• Gauge invariant operators in the CFT\textsubscript{4} are in one-to-one correspondence with fields (or extended objects) in AdS\textsubscript{5}.

• Operator dimension is determined by the mass of the dual field; e.g. for scalar operators

\[ \Delta_{\pm} = 2 \pm \sqrt{4 + m^2 L^2} \]

• Correlation functions are calculated from the dependence of string theory path integral on boundary conditions \( \phi_0 \) in AdS\textsubscript{5}, imposed near \( z=0 \):

\[ \langle \exp \int d^4x \phi_0 \mathcal{O} \rangle = Z_{\text{string}}[\phi_0] \]

• In the large N limit the path integral is found from the classical string action:

\[ Z_{\text{string}}[\phi_0] \sim \exp(-I[\phi_0]) \]
Conebrane Dualities

- To reduce the number of supersymmetries in AdS/CFT, we may place the stack of N D3-branes at the tip of a 6-d Ricci-flat cone X whose base is a 5-d Einstein space Y:
  \[ ds_X^2 = dt^2 + r^2 ds_Y^2 \]

- Taking the near-horizon limit of the background created by the N D3-branes, we find the space \( \text{AdS}_5 \times Y \), with N units of RR 5-form flux, whose radius is given by

- This type IIB background is conjectured to be dual to the IR limit of the gauge theory on N D3-branes at the tip of the cone X.

\[ L^4 = \frac{\sqrt{\pi \kappa N}}{2 \text{Vol}(Y)} = 4\pi g_s N \alpha'^2 \frac{\pi^3}{\text{Vol}(Y)} \]
Trace Anomaly

• In a 4-d CFT there are two trace anomaly coefficients, $a$ and $c$:

$$\langle T^\alpha_{\alpha} \rangle = -aE_4 - cI_4$$

• Calculations on AdS$_5 \times Y$ give their leading large $N$ values

Henningson, Skenderis; Gubser

• In super-conformal theories the anomalies are related to the spectrum of $R$-charges of the chiral fermions: Anselmi, Freedman, Grisaru, Johansen

$$E_4 = \frac{1}{16\pi^2} \left( R_{ijkl}^2 - 4R_{ij}^2 + R^2 \right)$$

$$I_4 = -\frac{1}{16\pi^2} \left( R_{ijkl}^2 - 2R_{ij}^2 + \frac{1}{3}R^2 \right)$$

$$a = c = \frac{\pi^2 N^2}{4 \text{Vol}(Y)}$$

$$a = \frac{3}{32} (3\text{Tr}R^3 - \text{Tr}R) ; \quad c = \frac{1}{32} (9\text{Tr}R^3 - 5\text{Tr}R)$$
• This provides basic checks of the dualities.

• For the $\mathcal{N}=4$ SYM theory the gauginos have $R=1$, while the fermion fields from the $\mathcal{Z}$ chiral multiplets have $R=-1/3$.

• Since $\text{Tr} R=0$, we find $a=c$, and

$$a = c = \frac{9}{32} \text{Tr} R^3 = (N^2 - 1) \frac{9}{32} (1 + 3(-1/3)^3) = \frac{N^2 - 1}{4}$$

• On the gravity side, the volume of $S^5$ is $\pi^3$ (the radius of $Y$ is fixed so that $R_{ij} = 4g_{ij}$)

• For large $N$ the two calculations of anomaly coefficients agree.
Orbifold Cones
Kachru, Silverstein; Lawrence, Nekrasov, Vafa

- The simplest set of examples is provided by cones that are orbifolds $\mathbb{R}^6/\Gamma$, where $\Gamma$ is a subgroup of the rotation group SU(4).
- For abelian orbifolds, all group elements can be brought to the form
- For $\mathbb{Z}_k$ orbifolds, the $n$-th group element is specified by three integers $m_i$ defined mod $k$: $x_i = nm_i/k$.
- If none of the eigenvalues of the generator = 1, then all SUSY is broken; if one of the eigenvalues = 1, then $\mathcal{N}=1$ SUSY is preserved; if two of the eigenvalues = 1, then $\mathcal{N}=2$ SUSY is preserved.
• The action in the n-th twisted sector on 3 complex coordinates of \( \mathbb{C}^3 \),
\[
Z^1 = X^1 + iX^2, \quad Z^2 = X^3 + iX^4, \quad Z^3 = X^5 + iX^6
\]
and their complex conjugates, is
\[
R(g_n) = \text{diag}(\omega_k^{n(m_1+m_2)}, \omega_k^{n(m_1+m_3)}, \omega_k^{n(m_2+m_3)}, \omega_k^{-n(m_1+m_2)}, \omega_k^{-n(m_1+m_3)}, \omega_k^{-n(m_2+m_3)})
\]
where \( \omega_k = e^{2\pi i/k} \).

• If none of these phases = 1, then the orbifold acts freely on \( S^5/\Gamma \). (The tip of the cone is a fixed point that is removed in the basic near-horizon limit.)

• A well-known example of a freely-acting orbifold is \( \mathbb{Z}_3 \) with \( m_i = 1 \). Since one of the eigenvalues of the generator = 1, i.e. \( \Gamma \subset SU(3) \), this orbifold preserves \( \mathcal{N}=1 \) SUSY.
Construction of the quiver gauge theories  Douglas, Moore

- Gauge theory on N D3-branes at the tip of $R^6/\Gamma$ is found by applying projections to the $U(Nk)$ gauge theory on the covering space. Retain only the fields invariant under the orbifold action combined with conjugation by a $U(Nk)$ matrix $\gamma$ acting on the gauge indices:

$$\gamma = \text{diag}(I_N, e^{2\pi i/k} I_N, e^{4\pi i/k} I_N, \ldots, e^{-2\pi i/k} I_N)$$

$$\psi^1 \rightarrow e^{2\pi im_1/k} \gamma \psi^1 \gamma^{-1}, \quad \psi^2 \rightarrow e^{2\pi im_2/k} \gamma \psi^2 \gamma^{-1}, \ldots$$

$$Z^1 \rightarrow e^{2\pi i(m_1+m_2)/k} \gamma Z^1 \gamma^{-1}, \quad Z^2 \rightarrow e^{2\pi i(m_1+m_3)/k} \gamma Z^2 \gamma^{-1}, \ldots$$
• In the supersymmetric examples, such as $C^3/Z_3$ or the conifold, the conebrane dualities have been tested almost as thoroughly as in the maximally supersymmetric case.

• But when all SUSY is broken, problems may arise.

• When the orbifold $\Gamma$ breaks all SUSY and is not freely acting, then the weakly curved background $\text{AdS}_5 \times S^5/\Gamma$ is unstable due to the presence of tachyons that have $(mL)^2 <-4$, and therefore violate the BF bound.

• But for freely acting orbifolds, the negative zero-point energy is compensated by the large stretching of the twisted sector closed strings in the compact space. Hence, at large radius, there are no `bad tachyons.' This makes freely acting orbifolds particularly interesting from AdS/CFT point of view.

• Yet, before the formal decoupling limit, the non-SUSY freely acting orbifolds have closed-string tachyons localized at the tip of the cone.
Closed String Zero-Point Energy

- In light-Cone Green-Schwarz, there are 4 complex world sheet bosons and fermions.
- In the n-th twisted sector the boundary conditions on the fermions are
  \[ b^l(\sigma + 2\pi) = e^{2\pi i n m_1 / k} b^l(\sigma) \]
  and on the bosons
- The total zero-point energy of these modes is
  \[
  8E_0(x_1, x_2, x_3) = -1 - (2\{x_1 + x_2\} - 1)^2 - (2\{x_1 + x_3\} - 1)^2 - (2\{x_2 + x_3\} - 1)^2 \\
  + (2\{x_1 + x_2 + x_3\} - 1)^2 + \sum_{i=1}^{3} (2\{x_i\} - 1)^2
  \]
Weak coupling analysis of non-SUSY quivers

• My recent work with Dymarsky and Roiban (hep-th/0505099 and 0509132) reconsiders quiver gauge theory on a stack of D3-branes at the tip of a cone $R^6/\Gamma$ where the orbifold group $\Gamma$ breaks all the supersymmetry.

• At first sight, the gauge theory seems conformal because the planar beta functions for all single-trace operators vanish. The candidate string dual is $\text{AdS}_5 \times S^5/\Gamma$. Kachru, Silverstein; Lawrence, Nekrasov, Vafa; Bershadsky, Johanson

• However, dimension 4 double-trace operators made out of twisted single-trace ones, $f O_n O_{-n}$, are induced at one-loop. Their planar beta-functions have the form

\[
\beta_f = a \lambda^2 + 2 \gamma f \lambda + f^2 \\
\beta_\lambda = 0
\]
A Note on Normalizations

- The VEV of a single trace operator is of order $N$.
- The standard Yang-Mills action is of order $N^2$ in the 't Hooft limit.
- The double-trace operators $f_{O_n O_{-n}}$ make contributions of the same order (for the coupling constant of order 1). They cannot be ignored in the leading large $N$ limit.
- In fact, the tree-level potential of the $SU(N)^k$ quiver theories (with the interacting $U(1)'s$ decoupled) contains such double-trace terms.

$$ S = - \int d^4x \frac{1}{2g_{YM}^2} \text{Tr} F_{\mu\nu}^2 $$
Problem 1

a) Derive the $Z_2$ quiver gauge theory obtained by the projection on the $U(2N) \mathcal{N} = 4$ SYM where the $Z_2$ is generated by $-I$ in $SU(4)$ accompanied by conjugation with $\gamma = \text{diag}(I_N, -I_N)$. What is the gauge group and field content? Is this a freely acting orbifold?

b) In this gauge theory, calculate the one-loop Coleman-Weinberg potential as a function of the eigenvalues of the adjoint scalars. What are the operators that pick up one-loop beta functions?
• If $D=\gamma^2 - a < 0$, then there is no real fixed point for $f$.

• A class of $\mathbb{Z}_k$ orbifolds with global SU(3) symmetry, that are freely acting on the 5-sphere, has the group action in the fundamental of SU(4)

$$r(g^n) = \text{diag}(\omega_k^n, \omega_k^m, \omega_k^n, \omega_k^{-3n})$$

$$\omega_k = e^{i\alpha_k}, \quad \alpha_k = \frac{2\pi}{k}$$

• Here is a plot of a one-loop SU(N)^k gauge theory discriminant, $D$, and of the ground state closed string $m^2$ on the cone without the D-branes. $n=1, \ldots, k-1$ labels the twisted sector, and $x=n/k$.

• The simplest freely acting non-susy example is $\mathbb{Z}_5$ where there are four induced double-trace couplings

$$\delta_2 \text{ trace } \mathcal{L} = f_{8,1} O_1^{(ij)} O_{-1}^{(ij)} + f_{8,2} O_2^{(ij)} O_{-2}^{(ij)} + f_{1,1} O_1 O_{-1} + f_{1,2} O_2 O_{-2}$$

• For example, the SU(3) adjoints

$$O_n^{(ij)}$$

are

$$\sum_{k=1}^{5} \left( \Phi_{i,k+2}^{j,k+2} \Phi_{i,j,k}^{j,k+2} \frac{1}{3} \eta^{ij} \Phi_{i,k+2}^{j,k+2} \Phi_{i,j,k}^{j,k+2} \right) e^{i\alpha(k-1)}$$

$(\alpha=2\pi/5)$
• For more complicated orbifolds, crossing of eigenvalues of the discriminant matrix becomes important. The agreement with closed strings continues to hold.

• Generally, there are three twist angles $x_i$ that define a cube. The stability/instability regions agree between one-loop gauge theory and string theory.
• Any non-SUSY abelian orbifold contains unstable operators. This appears to remove all such orbifold quivers from a list of large N perturbatively conformal gauge theories.

• The one-loop beta functions destroy the conformal invariance precisely in those twisted sectors where there exist closed-string tachyons localized at the tip of $\mathbb{R}^6/\Gamma$. Thus, a very simple correspondence emerges between perturbative gauge theory and free closed string on an orbifold. Why? Perhaps, in the presence of tachyons, the standard AdS/CFT decoupling argument may fail.

• The AdS$_5 \times S^5/\Gamma$ background is tachyon-free at large radius. Could it have some instabilities? If not, then there is a transition from instability to stability as $\lambda$ is increased.
• What is the end-point of the RG flow?

• Condensation of localized tachyon smoothes out the tip of the cone. Adams, Polchinski, Silverstein

• The gauge theory on D3-branes at a smooth point is $\mathcal{N}=4$ SYM. Hence, a natural conjecture is that the gauge theory flows from the non-SUSY $SU(N)^k$ quiver gauge theory to the $\mathcal{N}=4$ SU(N) SYM. Dymarsky, Franco, Roiban, IK (work in progress)
D3-branes on the Conifold

- The conifold is a Calabi-Yau 3-fold cone $X$ described by the constraint $\sum_{a=1}^{4} z_a^2 = 0$ on 4 complex variables.

- Its base $Y$ is a coset $T^{1,1}$ which has symmetry $SU(2)_A \times SU(2)_B$ that rotates the $z$’s, and also $U(1)_R$:

\[ z_a \rightarrow e^{i\theta} z_a \]

- The Sasaki-Einstein metric on $T^{1,1}$ is

\[
ds_{T^{1,1}}^2 = \frac{1}{9} \left( d\psi + \sum_{i=1}^{2} \cos \theta_i d\phi_i \right)^2 + \frac{1}{6} \sum_{i=1}^{2} \left( d\theta_i^2 + \sin^2 \theta_i d\phi_i^2 \right) \]

where $\theta_i \in [0, \pi]$, $\phi_i \in [0, 2\pi]$, $\psi \in [0, 4\pi]$.

- The topology of $T^{1,1}$ is $S^2 \times S^3$. 
• The $\mathcal{N}=1$ SCFT on the D3-branes at the apex of the conifold has gauge group $SU(N) \times SU(N)$ coupled to bifundamental chiral superfields $A_1, A_2$, in $(\bar{N}, N)$, and $B_1, B_2$ in $(N, \bar{N})$. I.K, Witten

• The R-charge of each fields is $1/2$. This insures $U(1)_R$ anomaly cancellation.

• The unique $SU(2)_A \times SU(2)_B$ invariant, exactly marginal quartic superpotential is added:

$$W = \epsilon^{ij} \epsilon^{kl} \text{tr} A_i B_k A_j B_l$$

• This theory also has a baryonic $U(1)$ symmetry under which $A_k \rightarrow e^{i\alpha} A_k$; $B_l \rightarrow e^{-i\alpha} B_l$, and a $Z_2$ symmetry which interchanges the A’s with the B’s and implements charge conjugation.
Comparison with a $\mathbb{Z}_2$ Orbifold Quiver

- The simplest $\mathcal{N}=2$ SUSY quiver has $k=2$; $m_1=m_2=1$, $m_3=0$. The gauge group is again $\text{SU}(N) \times \text{SU}(N)$, but in addition to the bifundamentals $A_i$, $B_j$, there is one adjoint chiral superfield for each gauge group, with superpotential $g\text{Tr}\Phi(A_1B_1 - A_2B_2) + g\text{Tr}\tilde{\Phi}(B_1A_1 - B_2A_2)$

- Adding a $\mathbb{Z}_2$ odd mass term and integrating out the adjoints, we obtain the superpotential of the conifold theory, $\frac{m}{2}(\text{Tr}\Phi^2 - \text{Tr}\tilde{\Phi}^2)$
Problem 2

In a supersymmetric field theory, the trace anomaly coefficients $a$ and $c$ are given by the formulae

$$a = \frac{3}{32} \left( 3 \text{Tr} R^3 - 3 \text{Tr} R \right), \quad c = \frac{1}{32} \left( 9 \text{Tr} R^3 - 5 \text{Tr} R \right),$$

where $R$ refers to the $U(1)_R$ charges, and the trace is over all the chiral fermion fields.

a) Calculate $a$ and $c$ in the following two gauge theories: the $\mathcal{N} = 2$ supersymmetric $Z_2$ orbifold quiver, and in the $\mathcal{N} = 1$ SCFT on $N$ D3-branes at the conifold.

b) For $AdS_5 \times Y$ with $N$ units of RR 5-form flux, it was found at leading order in $N$ that

$$a = c = \frac{N^2 \pi^3}{4 \text{vol}(Y)},$$

where the radius of $Y$ is normalized so that $R_{ij} = 4g_{ij}$ on $Y$. Compare this formula with the gauge theory results of part a).
Breaking the Conformal Symmetry

• A useful tool is to add to the N D3-branes M D5-branes wrapped over the $S^2$ at the tip of the conifold.

• The 10-d geometry dual to the gauge theory on these branes is the warped deformed conifold (IK, Strassler)

$$ds_{10}^2 = h^{-1/2}(t)(- (dx^0)^2 + (dx^i)^2) + h^{1/2}(t)ds_6^2$$

• $ds_6^2$ is the metric of the deformed conifold, a simple Calabi-Yau space defined by the following constraint on 4 complex variables:

$$\sum_{i=1}^{4} z_i^2 = \varepsilon^2$$
String Theoretic Approach to Confinement

- It is possible to generalize the AdS/CFT correspondence in such a way that the quark-antiquark potential is linear at large distance.
- A "cartoon" of the necessary metric is

\[ ds^2 = \frac{dz^2}{z^2} + a^2(z)(- (d\tilde{x}^0)^2 + (d\tilde{x}^i)^2) \]

- The space ends at a maximum value of \( z \) where the warp factor is finite. Then the confining string tension is

\[ a^2(z_{\text{max}}) \]

\[ \frac{2\pi \alpha'}{2\pi \alpha'} \]
• The warp factor is finite at the `end of space’ t=0, as required for the confinement: \( h(t) = 2^{-8/3} \gamma I(t) \)

\[
I(t) = \int_t^\infty dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh 2x - 2x)^{1/3} , \quad \gamma = 2^{10/3} (g_s M \alpha')^2 \varepsilon^{-8/3}
\]

• The standard warp factor \( a^2 \), which measures the string tension, is identified with \( h(t)^{-1/2} \) and is minimized at \( t=0 \). It blows up at large \( t \) (near the boundary).

• The dilaton is exactly constant due to the self-duality of the 3-form background

\[
*_6 G_3 = iG_3 , \quad G_3 = F_3 - \frac{i}{g_s} H_3
\]
• The radius-squared of the $S^3$ at $t=0$ is $g_sM$ in string units.

• When $g_sM$ is large, the curvatures are small everywhere, and the SUGRA solution is reliable in "solving" this confining gauge theory.
Anomalously light glueballs

- The confining string tension is
  \[ T_s = \frac{1}{2^{4/3} \alpha_0^{1/2} \pi (\alpha')^2 g_s M} \]

- The glueballs are the normalizable modes localized near at small \( t \). In the supergravity limit (at large \( g_s M \)) their mass scales are

  \[ m_{\text{glueball}} \sim m_{KK} \sim \frac{\varepsilon^{2/3}}{g_s M \alpha'} \]

  \[ T_s \sim g_s M (m_{\text{glueball}})^2 \]

- In order to eliminate the anomalously light bound states, we need a small \( g_s M \), which requires a departure from the SUGRA limit.

- Even for small \( g_s M \), SUGRA becomes reliable in the UV (at large \( t \)).
Log running of couplings

• The large radius asymptotic solution is characterized by logarithmic deviations from $\text{AdS}_5 \times T^{1,1}$, Tseytlin

• The near-AdS radial coordinate is

$\rho \sim \varepsilon^{2/3} e^{t/3}$

• The NS-NS and R-R 2-form potentials:

$$F_3 = \frac{M \alpha'}{2} \omega_3, \quad B_2 = \frac{3 g_s M \alpha'}{2} \omega_2 \ln(r/r_0)$$

$$\omega_2 = \frac{1}{2} (g^1 \wedge g^2 + g^3 \wedge g^4) = \frac{1}{2} (\sin \theta_1 d \theta_1 \wedge d \phi_1 - \sin \theta_2 d \theta_2 \wedge d \phi_2)$$

$$\omega_3 = \frac{1}{2} g^5 \wedge (g^1 \wedge g^2 + g^3 \wedge g^4)$$
• This translates into log running of the gauge couplings through

\[
\frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2} = \frac{\pi}{g_s e^\Phi},
\]

\[
\left[\frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_2^2}\right] g_s e^\Phi = \frac{1}{2\pi\alpha'} \left(\int_{S^2} B_2\right) - \pi
\]

• The warp factor deviates from the M=0 solution logarithmically.

\[
h(r) = \frac{27\pi(\alpha')^2 [g_s N + a(g_s M)^2 \ln(r/r_0) + a(g_s M)^2/4]}{4r^4}
\]

• Remarkably, the 5-form flux, dual to the number of colors, also changes logarithmically with the RG scale.

\[
\tilde{F}_5 = F_5 + *F_5, \quad F_5 = 27\pi\alpha'^2 N_{eff}(r) \text{vol}(T^{1,1})
\]

\[
N_{eff}(r) = N + \frac{3}{2\pi} g_s M^2 \ln(r/r_0)
\]
• What is the explanation in the dual SU(kM) x SU((k-1)M) SYM theory coupled to bifundamental chiral superfields $A_1$, $A_2$, $B_1$, $B_2$? A novel phenomenon, called a duality cascade, takes place: $k$ repeatedly changes by 1 as a result of the Seiberg duality. 

IK, Strassler

(diagram of RG flows from a review by M. Strassler)
Y_{p,q} Dualities

- Cascading behavior is not limited to the conifold. Recently, an infinite family of new CY cones over Sasaki-Einstein spaces Y_{p,q} of topology S^2 \times S^3 have been constructed (p and q are co-prime integers).

\[ d\Omega^2_{Y_{p,q}} = (e^\theta)^2 + (e^\phi)^2 + (e^y)^2 + (e^\beta)^2 + (e^\psi)^2 \]

\[ e^\theta = \sqrt{\frac{1-y}{6}} \, d\theta \, , \quad e^\phi = \sqrt{\frac{1-y}{6}} \, \sin \theta d\phi \]
\[ e^y = \frac{1}{\sqrt{wv}} dy \, , \quad e^\beta = \frac{\sqrt{wv}}{6} (d\beta + \cos \theta d\phi) \]
\[ e^\psi = \frac{1}{3} (d\psi - \cos \theta d\phi + y (d\beta + \cos \theta d\phi)) \]

\[ w(y) = \frac{2(b - y^2)}{1-y} \, , \quad v(y) = \frac{b - 3y^2 + 2y^3}{b - y^2} \]

\[ b = \frac{1}{2} - \frac{p^2 - 3q^2}{4p^3} \sqrt{4p^2 - 3q^2} \]

Gauntlett, Martelli, Sparks, Waldram
• y ranges between two smaller roots of $v(y)$:

$$y_1 = \frac{1}{4p} \left(2p - 3q - \sqrt{4p^2 - 3q^2}\right)$$

$$y_2 = \frac{1}{4p} \left(2p + 3q - \sqrt{4p^2 - 3q^2}\right)$$

$$\text{Vol}(Y^{p,q}) = \frac{q^2 \left(2p + \sqrt{4p^2 - 3q^2}\right)}{3p^2 \left(3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}\right)} \pi^3$$

• The $SU(N)^{2p}$ SCFT’s on N D3-branes at the tip of the cones have also been constructed. Benvenuti, Franco, Hanany, Martelli, Sparks

• For example, here is the quiver diagram for the SCFT dual to $AdS_5 \times Y^{4,3}$
R-charges from a-maximization

• The conformal invariance conditions do not fully determine the R-charges. Let
  \( R_Z = x, \ R_Y = y, \ R_U = 1 - (x+y)/2, \ R_V = 1 + (x-y)/2 \)

• The technique of a-maximization [Intriligator, Wecht] gives

\[
\begin{align*}
  x &= \frac{1}{3q^2} \left( -4p^2 + 2pq + 3q^2 + (2p - q)\sqrt{4p^2 - 3q^2} \right) \\
  y &= \frac{1}{3q^2} \left( -4p^2 - 2pq + 3q^2 + (2p + q)\sqrt{4p^2 - 3q^2} \right)
\end{align*}
\]

• Remarkably, this gives the trace anomaly agreeing with the AdS/CFT

\[
a(Y^{p,q}) = \frac{\pi^3 N^2}{4 \text{Vol}(Y^{p,q})}
\]

Benvenuti et al; Bertolini, Bigazzi, Cotrone
• Addition of M D5-branes wrapped over the $S^2$ modifies the quiver diagram to

• Performing the Seiberg duality on the biggest gauge group gives the same quiver with $N \rightarrow N-M$. This fact is necessary for the existence of a self-similar cascading RG flow.

• The gravity duals of these cascades include the ISD 3-form field strength. Herzog, Ejaz, IK

\[
\Omega_{2,1} = K \left( \frac{dr}{r} + ie^\psi \right) \wedge \omega
\]

\[
\omega = F(y) (e^\theta \wedge e^\phi - e^y \wedge e^z)
\]

\[
F(y) = \frac{1}{(1 - y)^2}
\]
The metric and 5-form are determined by a single warp factor, which however depends on 2 coordinates.

Luckily, the PDE for $h(r,y)$ is exactly solvable.

The log behavior characteristic of the cascade produces a naked singularity in the IR. Is there a smooth solution with the above large $r$ behavior?

\[ ds^2 = h^{-1/2} dx_4^2 + h^{1/2} (dr^2 + r^2 d\Omega^2_{Y_{\nu,\alpha}}) \]

\[ g_5 F_5 = d(h^{-1}) \wedge d^4 x + *[d(h^{-1}) \wedge d^4 x] \]

\[ h(r, y) = \frac{A \ln(r/r_0) + s(y)}{r^4} \]

\[ s(y) = -\frac{C}{4(b-1)} \left[ \frac{1}{1-y} + \frac{(1+2y_1)(1+2y_2) \ln(y_3 - y)}{2(b-1)} \right] \]
IR Behavior of the Conifold Cascade

- Here the dynamical deformation of the conifold renders the solution smooth, and explains the IR dynamics of the gauge theory.

- **Dimensional transmutation** in the IR. The dynamically generated confinement scale is
  \[ \sim \varepsilon^{2/3} \]

- The pattern of **R-symmetry breaking** is the same as in the SU(M) SYM theory: \( Z_{2M} \rightarrow Z_2 \)

- Yet, the IR gauge theory is somewhat more complicated.
• In the IR the gauge theory cascades down to SU(2M) x SU(M). The SU(2M) gauge group effectively has $N_f=N_c$.

• The baryon and anti-baryon operators acquire expectation values and break the U(1) symmetry under which $A_k \rightarrow e^{\imath a} A_k; \ B_l \rightarrow e^{-\imath a} B_l$. Hence, we observe confinement without a mass gap: due to $U(1)_{baryon}$ chiral symmetry breaking there exist a Goldstone boson and its massless scalar superpartner. There exists a baryonic branch of the moduli space

\[
\begin{align*}
\mathcal{A} &= \epsilon^{i_1 \ldots i_{N_c}} A^{a_{i_1}}_{\alpha_{i_1}} \cdots A^{a_{i_{N_c}}}_{\alpha_{N_c}} \\
\mathcal{B} &= \epsilon_{i_1 \ldots i_{N_c}} B^{i_{i_1}}_{\dot{\alpha}_{i_1}} \cdots B^{i_{i_{N_c}}}_{\dot{\alpha}_{N_c}}
\end{align*}
\]

\[
\begin{align*}
A &= i\Lambda_1^{2M} \zeta \\
B &= i\Lambda_1^{2M} / \zeta
\end{align*}
\]
• The KS solution is part of a moduli space of confining SUGRA backgrounds, resolved warped deformed conifolds. Gubser, Herzog, IK; Butti, Grana, Minasian, Petrini, Zaffaroni

• To look for them we need to use the PT ansatz:

\[
\begin{align*}
 ds_{10}^2 &= H^{-1/2} dx_m dx_m + e^x ds_6^2, \\
 ds_6^2 &= (e^g + a^2 e^{-g})(c_1^2 + c_2^2) + e^{-g} \sum_{i=1}^2 (c_i^2 - 2ae_i e_i) + v^{-1}(c_3^2 + dt^2)
\end{align*}
\]

• H, x, g, a, v, and the dilaton are functions of the radial variable t.

• Additional radial functions enter into the ansatz for the 3-form field strengths. The PT ansatz preserves the SO(4) but breaks a \( \mathbb{Z}_2 \) charge conjugation symmetry, except at the KS point.
• BGMPZ used the method of SU(3) structures to derive the complete set of coupled first-order equations.

• A result of their integration is that the warp factor and the dilaton are related:

\[ H(t) = \tilde{H} \left( e^{-2\phi(t)} - 1 \right) \]

Dymarsky, IK, Seiberg

• The integration constant determines the \`modulus\' U:

\[ \tilde{H} = \gamma U^{-2} \quad \text{where} \quad \gamma = 2^{10/3} (g_s M \alpha')^2 \varepsilon^{-8/3} \]

• At large \( t \) the solution approaches the KT \`cascade asymptotics\':

\[ a(t) = -2e^{-t} + U e^{-5t/3}(-t + 1) + \ldots \]

\[ \gamma^{-1} H(t) = \frac{3}{32} e^{-4t/3} (4t - 1) - \frac{3}{32 \cdot 512} U^2 (256t^3 - 864t^2 + 1752t - 847)e^{-8t/3} + O \left( e^{-10t/3} \right) \]
• The resolution parameter \( U \) is proportional to the VEV of the operator 
\[
U = \text{Tr} \left( \sum_{\alpha} A_{\alpha} A_{\alpha}^\dagger - \sum_{\dot{\alpha}} B_{\dot{\alpha}} B_{\dot{\alpha}}^\dagger \right)
\]

• This family of resolved warped deformed conifolds is dual to the \`baryonic branch\’ in the gauge theory (the quantum deformed moduli space).

• At large \( U \) the IR part of the solution approaches that of the Maldacena-Nunez solution. But we always have the \`cascade\’ asymptotics at large \( t \).

• Here are plots of the string tension (a fundamental string at the bottom of the throat is dual to an \`emergent\’ chromo-electric flux tube) and of the dilaton profiles as a function of the modulus \( U = \ln |\zeta| \). Dymarsky, IK, Seiberg
BPS Domain Walls

- A D5-brane wrapped over the 3-sphere at the bottom of the throat is the domain wall separating two adjacent vacua of the theory.
- Since it is BPS saturated, its tension cannot depend on the baryonic branch modulus. This is indeed the case. This fact provides a check on the choice of the UV boundary conditions, and on the numerical integration procedure necessary away from the KS point.
- Analytic proof?
Applications to D-brane Inflation

- The Slow-Roll Inflationary Universe (Linde; Albrecht, Steinhardt) is a very promising idea for generating the CMB anisotropy spectrum observed by the WMAP.

- Finding models with very flat potentials has proven to be difficult. Recent string theory constructions use moving D-branes. Dvali, Tye, ...

- In the KKLT/KKLMMT model, the warped deformed conifold is embedded into a string compactification. An anti-D3-brane is added at the bottom to break SUSY and generate a potential. A D3-brane rolls in the throat. Its radial coordinate plays the role of an inflaton.

Kachru, Kallosh, Linde, Maldacena, McAllister, Trivedi
A related suggestion for D-brane inflation (A. Dymarsky, I.K., N. Seiberg)

- In a flux compactification, the $U(1)_{\text{baryon}}$ is gauged. Turn on a Fayet-Iliopoulos parameter $\xi$.
- This makes the throat a resolved warped deformed conifold.
- The probe D3-brane potential on this space is asymptotically flat, if we ignore effects of compactification and D7-branes. The plots are for two different values of $U\sim \xi$.
- No anti-D3 needed: in presence of the D3-brane, SUSY is broken by the D-term $\xi$. Related to the `D-term Inflation’ Binétruy, Dvali; Halyo
Slow roll D-brane inflation?

- Effects of D7-branes and of compactification generically spoil the flatness of the potential. Non-perturbative effects introduce the KKLT-type superpotential

\[ W = W_0 + A(X)e^{-a\rho} \]

where \( X \) denotes the D3-brane position. In any warped throat D-brane inflation model, it is important to calculate \( A(X) \).
• The gauge theory on D7-branes wrapping a 4-cycle $\Sigma_4$ has coupling

\[ \frac{1}{g^2} = \frac{V_{\Sigma_4}^w}{g_7^2} = \frac{T_3 V_{\Sigma_4}^w}{8\pi^2} \]

• The non-perturbative superpotential depends on the D3-brane location through the warped volume

\[ V_{\Sigma_4}^w = \int_{\Sigma_4} d^4 \xi \sqrt{g^{i\alpha\beta} h(X)} \]

• In the throat approximation, the warp factor can be calculated and integrated over a 4-cycle explicitly. Baumann, Dymarsky, Klebanov, Maldacena, McAllister, Murugan.

• For a class of conifold embeddings

\[ \prod_{i=1}^{4} w_i^{p_i} = \mu^P \]

\[ P \equiv \sum_{i=1}^{4} p_i \]

the result is

\[ A = A_0 \left( \frac{\mu^P - \prod_{i=1}^{4} w_i^{p_i}}{\mu^P} \right)^{1/n} \]
• This formula applies both to $n$ wrapped D7-branes, and to a wrapped Euclidean D3 ($n=1$).

• For the latter case, Ganor showed that $A$ has a simple zero when the D3-brane approaches the 4-cycle. Our result agrees with this.

• We have also carried out such calculations for 4-cycles within the Calabi-Yau cones over $Y^{p,q}$ with analogous results: $A(X)$ is proportional to the embedding equation raised to the power $1/n$. This appears to be a general rule for 4-cycles in the throat.
• The dependence of the non-perturbative superpotential on D3-brane position, and other compactification effects, give Hubble-scale corrections to the inflaton potential.

• Some `fine-tuning’ is generally needed to cancel different corrections to the D3-brane potential. This is currently under investigation with D. Baumann, A. Dymarsky, J. Maldacena, L. McAllister and P. Steinhardt.
Conclusions

• In the first part, we investigated non-SUSY orbifolds of AdS/CFT. At one loop, flow of double-trace couplings spoils conformal invariance even in the large N limit. There is a precise connection of this instability with presence of twisted sector closed string tachyons.

• Gauge/string dualities for confining gauge theories give a new geometrical view of such important phenomena as dimensional transmutation, chiral symmetry breaking, and quantum deformation of moduli space. We have also discussed new UV phenomena: the duality cascades.

• Embedding gauge/string dualities into string compactifications offers new possibilities for modeling inflation. In particular, D3-branes on resolved warped deformed conifolds may realize D-term inflation.

• Calculation of non-perturbative corrections to the inflaton potential is important for determining if these models can produce slow-roll inflation.