Think Globally, Act Locally

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Quantum Fields beyond Perturbation Theory, KITP 2014

Ofer Aharony, NS, Yuji Tachikawa, arXiv:1305.0318
Anton Kapustin, NS, arXiv:1401.0740
Seemingly unrelated questions

1. 2d Ising has two phases:
   1. High T, unbroken global $\mathbb{Z}_n$
   2. Low T, broken global $\mathbb{Z}_n$, $n$ vacua

   Duality should exchange them. How can this be consistent?

   Similar question for the 4d $\mathbb{Z}_n$ lattice gauge theory

2. In $su(n)$ gauge theory ‘t Hooft and Wilson operators satisfy the equal time (‘t Hooft) commutation relations:

   $$W(\gamma) T(\gamma') = e^{2\pi i \ell(\gamma,\gamma')/n} T(\gamma') W(\gamma)$$

   They are spacelike separated, so how can they fail to commute?
Seemingly unrelated questions

3. The de-confinement transition of a 4d $SU(n)$ gauge theory is related to a global $\mathbb{Z}_n$ symmetry in 3d. What is its 4d origin? Can we gauge it?

4. The partition function $\mathcal{Z} = \sum c_k \mathcal{Z}_k$ is a sum of contributions from distinct topological sectors (e.g. twisted boundary conditions in orbifolds, different instanton numbers in a gauge theory).
   
   1. What are the physical restrictions on $c_k$?
   
   2. Are there new theories with different values of $c_k$ (e.g. restrict the sum over the instanton number)?

5. S-duality in $\mathcal{N} = 4$ SUSY maps $SU(2) \leftrightarrow SO(3)$. How is this consistent with $(ST)^3 = 1$?

6. Can we relate $SU(n)$ and $SU(n)/\mathbb{Z}_n$ gauge theories as in orbifolds?
Coupling a QFT to a TQFT

Unified framework:
Couple an ordinary quantum field theory to a topological theory.

In many cases such a coupling affects the local structure, e.g.:
• Free matter fields coupled to a Chern-Simons theory in $3d$.
• Orbifolds in $2d$ CFT

Often the local structure is not affected, but there are still interesting consequences: spectrum of line and surface operators, local structure after compactification...
Outline

• Line operators
• Higher form global symmetries and their gauging
• Review of a simple TFT – a $4d \mathbb{Z}_n$ gauge theory
• $SU(n)/\mathbb{Z}_n$ gauge theories
• Modifying the sum over topological sectors (constraining the instanton number – restricting the range of $\theta$)
• Topological $\mathbb{Z}_n$ lattice gauge theory
• Turning an $SU(n)$ lattice gauge theory to an $SU(n)/\mathbb{Z}_n$ theory
• Duality in $2d$ Ising and $4d \mathbb{Z}_n$ lattice gauge theory

• Answering the seemingly unrelated questions
Line operators

Some line operators are boundaries of surfaces.
1. If the results depend on the geometry of the surface, this is not a line operator.
2. In some cases the dependence on the surface is only through its topology.
3. Genuine line operators are independent of the surface.

\[ W(\gamma) \mathcal{T}(\gamma') = e^{2\pi i \ell(\gamma, \gamma')/n} \mathcal{T}(\gamma') W(\gamma) \]

Here at least one of the line operators needs a surface. Hence, the apparent lack of locality.

Genuine line operators of the form \[ W(\gamma)^n \mathcal{T}(\gamma)^m \] with appropriate \( n \) and \( m \) are relatively local.
Higher-form global symmetries

Continuous $q$-form global symmetry – transformation with a closed $q$-form $\epsilon^{(q)}$ ($q = 0$ is an ordinary global symmetry with constant $\epsilon$).

Discrete $q$-form global symmetry $\int \epsilon^{(q)} \in 2\pi\mathbb{Z}$.

Example:
An ordinary gauge theory with group $G$ is characterized by transition functions $g_{ij} \in G$ with $g_{ij}g_{jk}g_{ki} = 1$.
If no matter fields transforming under $\mathbb{C}$, the center of $G$, $1$-form discrete global symmetry $g_{ij} \rightarrow h_{ij}g_{ij}$ with $h_{ij} \in \mathbb{C}$ and $h_{ij}h_{jk}h_{ki} = 1$. 
Higher-form global symmetries

1-form discrete global symmetry $g_{ij} \rightarrow h_{ij} g_{ij}$ with $h_{ij} \in \mathbb{C}$ and $h_{ij} h_{jk} h_{ki} = 1$.

When compactified on a circle, this 1-form global symmetry leads to an ordinary global symmetry $\mathbb{C}$.

It is common in thermal physics – the Polyakov loop is the order parameter for its breaking.

We gauge $\mathbb{C}$ by relaxing $h_{ij} h_{jk} h_{ki} = 1$ (analog of gauging an ordinary global symmetry by letting $\epsilon$ depend on position).

The resulting theory is an ordinary gauge theory of $G/\mathbb{C}$. 
A simple TFT – a $4d \mathbb{Z}_n$ gauge theory

[Maldacena, Moore, NS; Banks, NS]

1. Can describe as a $\mathbb{Z}_n$ gauge theory.
2. Can introduce a compact scalar $\phi \sim \phi + 2\pi$ and a $U(1)$ gauge symmetry $\phi \rightarrow \phi + n\lambda$ (with $\lambda \sim \lambda + 2\pi$).
3. Can also introduce a $U(1)$ gauge field $A$ with Lagrangian

$$\frac{i}{2\pi} H \wedge (d\phi + nA)$$

$H$ is a 3-form Lagrange multiplier. $U(1) \rightarrow \mathbb{Z}_n$ manifest.
4. Can dualize $\phi$ to find

$$\frac{in}{2\pi} B \wedge F$$

with $F = dA$ and $H = dB$. 
The basic TFT – a $4d \mathbb{Z}_n$ gauge theory

$$\frac{in}{2\pi} B \wedge F$$

5. Can dualize $A$ to find

$$\frac{i}{2\pi} F \wedge (d\hat{A} + nB)$$

$F$ is a 2-form Lagrange multiplier.

Gauge symmetry:

$$\hat{A} \rightarrow \hat{A} + d\hat{\lambda} - n\Lambda$$

$$B \rightarrow B + d\Lambda$$

6. Can keep only $\hat{A}$ with its gauge symmetry

7. Locally, can fix the gauge $\hat{A} = 0$ and have only a $\mathbb{Z}_n$ 1-form gauge symmetry
Observables in a $4d \mathbb{Z}_n$ gauge theory

$$\frac{i n}{2\pi} B \wedge F$$

Two kinds of Wilson operators

$$W_A(\gamma) = e^{i \oint_{\gamma} A}$$

$$W_B(\Sigma) = e^{i \oint_{\Sigma} B}$$

Their correlation functions

$$\langle W_B(\Sigma) W_A(\gamma) \rangle = e^{2\pi i \ell(\Sigma, \gamma)/n}$$

No additional (‘t Hooft) operators using $\hat{A}$ or $\phi$ – they are trivial.
An added term in a $4d \mathbb{Z}_n$ gauge theory

$$\frac{i}{2\pi} F \wedge (d\hat{A} + nB) \quad \quad \frac{in}{2\pi} B \wedge F$$

In any of the formulations we can add the term [Gukov, Kapustin; Kapustin, Thorngren]

$$\frac{ipn}{4\pi} B \wedge B = \frac{ip}{4\pi n} d\hat{A} \wedge d\hat{A}$$

With the modified gauge transformations:

$$\hat{A} \rightarrow \hat{A} + d\hat{\lambda} - n\Lambda$$

$$B \rightarrow B + d\Lambda$$

$$A \rightarrow A - p\Lambda$$

Consistency demands $\frac{pn}{2} \in \mathbb{Z}$ and $p \sim p + 2n$. 
Variants of the $4d \mathbb{Z}_n$ gauge theory

An obvious generalization is to use $q$ and $(D - q - 1)$-form gauge fields in $D$ dimensions

$$\frac{in}{2\pi} A^{(q)} \wedge dA^{(D-q-1)}$$

This is particularly interesting for $q = 0$ (or $q = D - 1$)

$$\frac{in}{2\pi} \Phi \wedge dA^{(D-1)} = \frac{in}{2\pi} \Phi \wedge F^{(D)}$$

Low energy theory of a system with a global $\mathbb{Z}_n$ symmetry. The order parameter for the symmetry breaking is $e^{i\Phi}$. The Wilson operator

$$W_A(\Sigma) = \exp \left( i \oint_{\Sigma} A^{(D-1)} \right)$$

describes a domain wall between different vacua.
From $SU(n)$ to $SU(n)/\mathbb{Z}_n$

$G = SU(n)$ gauge theory

- The Wilson line $W$ is a genuine line operator
- The ‘t Hooft line $\mathcal{T}$ needs a surface (the Dirac string). Hence the nonlocality in the commutation relations

$$W(\gamma)\mathcal{T}(\gamma') = e^{2\pi i\ell(\gamma,\gamma')/n}\mathcal{T}(\gamma')W(\gamma)$$

- If no matter fields charged under the $\mathbb{Z}_n$ center
  - Only the topology of the surface is important. (Like disorder operator in the Ising model with vanishing magnetic field.)
  - Global 1-form discrete symmetry $C = \mathbb{Z}_n$
From $SU(n)$ to $SU(n)/\mathbb{Z}_n$ gauge theories

Next, we gauge the 1-form symmetry $\mathcal{C} = \mathbb{Z}_n$ to find an $SU(n)/\mathbb{Z}_n$ g.t.
Can use any of the formulations of a $\mathcal{C} = \mathbb{Z}_n$ gauge theory.
Extend $SU(n)$ to $U(n)$ by adding $\hat{A}$ and impose the 1-form gauge symmetry $\hat{A} \rightarrow \hat{A} - n \Lambda$ to remove the added local dof.
We can also add a new term in this theory – a discrete $\theta$-term
\[
\frac{ip}{4\pi n} d\hat{A} \wedge d\hat{A} + \frac{pn}{2} \in \mathbb{Z} \quad p \sim p + 2n
\]
(Interpreted as an $SU(n)/\mathbb{Z}_n$ theory, it is identified with the Pontryagin square $w_2^2$ of the gauge bundle [Aharony, NS, Tachikawa].
Here, a manifestly local expression for it.)
\(SU(n)/\mathbb{Z}_n\) gauge theory – operators

Use e.g.

\[
\frac{i}{2\pi} F \wedge (d\hat{A} + nB) + \frac{ipn}{4\pi} B \wedge B \quad \frac{pn}{2} \in \mathbb{Z}
\]

\[
B \rightarrow B + d\Lambda
\]

\[
\hat{A} \rightarrow \hat{A} - n\Lambda
\]

\[
F \rightarrow F - pd\Lambda
\]

The surface operator

\[
e^{i \oint_\Sigma B} = e^{-\frac{i}{n} \oint_\Sigma d\hat{A}}
\]

measures the ‘t Hooft magnetic flux (\(w_2\) of the bundle) through \(\Sigma\).

It is a manifestly local expression – integral of a local density.

(More complicated expression for torsion cycles.)
\textbf{SU}(n)/\mathbb{Z}_n \text{ gauge theory}

\begin{align*}
\text{Wilson} & \quad W(\gamma) = \left( W_{f}^{SU(n)}(\gamma)e^{\frac{i}{n}\oint_{\gamma} \hat{A}} \right) e^{i \int_{\Sigma} B} \quad \partial \Sigma = \gamma \\
\text{‘t Hooft} & \quad \mathcal{T}(\gamma) = e^{i \oint_{\gamma} A + ip \int_{\Sigma} B}
\end{align*}

The dependence on \( \Sigma \) is topological.

Genuine line operators (dyonic) \( \mathcal{T}(\gamma)W(\gamma)^{-p} \)

The parameter \( p \) is a discrete \( \theta \)-parameter [Aharony, NS, Tachikawa].

- It labels distinct \( SU(n)/\mathbb{Z}_n \) theories.
- It can be understood either as a new term in the Lagrangian
  \( \frac{ip}{4\pi n} d \hat{A} \wedge d \hat{A} \) (it is the Pontryagin square \( w_2^2 \) of the gauge bundle), or in terms of the choice of genuine line operators.
Restricting the range of the $\theta$-angle [NS 2010]

- Similarly, we can restrict the range of $\theta$ by coupling a standard gauge theory to a $\mathbb{Z}_n$ topological gauge theory (the one related to a broken global $\mathbb{Z}_n$ symmetry)

\[
\frac{i n}{2\pi} \Phi F^{(4)} \quad \text{with} \quad \Phi \sim \Phi + 2\pi \quad ; \quad \int F^{(4)} \in 2\pi \mathbb{Z}
\]

\[
\cdots + \frac{i \theta}{16\pi^2} \text{Tr} F \tilde{F} + \frac{i n}{2\pi} \Phi F^{(4)} + \frac{i \Phi}{16\pi^2} \text{Tr} F \tilde{F}
\]

- The integral over $\Phi$ forces the topological charge to be a multiple of $n$. Hence, $\theta \sim \theta + 2\pi/n$.

- $\theta$ and $\theta + 2\pi/n$ are in the same superselection sector.

- $\Phi$ is a “discrete axion.”

- Note, this is consistent with locality and clustering!
Lattice gauge theory

- The variables of a lattice gauge theory are group elements on the links $U_l$. The gauge symmetry acts on the sites and the gauge invariant interaction is in terms of products around the plaquettes $U_p = \text{Tr} (\prod_l U_l)$.

- For a $\mathbb{Z}_n$ gauge theory we write $U_l = \exp(2\pi i \, u_l/n)$ and $U_p = \exp(2\pi i \, u_p/n)$.

- A 1-form gauge symmetry (Kalb-Ramond) resides on the links with gauge fields on the plaquettes. Such a $\mathbb{Z}_n$ gauge theory has variables $V_p = \exp(2\pi i \, v_p/n)$ and the gauge invariant variables on the cubes are $V_c = \exp(2\pi i \, v_c/n)$. 
A topological $\mathbb{Z}_n$ gauge theory is based on $U_l = \exp(2\pi i u_l/n)$ on the links and the Boltzmann weight

$$\prod_p U_p^{v_p} = \prod_p e^{2\pi i u_p v_p / n}$$

with $u_p$ are derived from $u_l$. $V_p = \exp(2\pi i v_p / n) \in \mathbb{Z}_n$ are gauge fields on the dual of $p$. They impose the constraint $U_p = \exp(2\pi i u_p / n) = 1$.

- This theory differs from the ordinary $\mathbb{Z}_n$ lattice gauge theory by this flatness constraint.
- Note the similarity to $BF$-theories.
- Easy to generalize to higher form gauge theories.
From $SU(n)$ to $SU(n)/\mathbb{Z}_n$ on the lattice

Starting with an $SU(n)$ lattice gauge theory with $U_l \in SU(n)$ construct an $SU(n)/\mathbb{Z}_n$ gauge theory [Halliday, Schwimmer]. An $SU(n)$ lattice gauge theory has a global 1-form symmetry $U_l \rightarrow h_l U_l$ with $h_l \in \mathbb{Z}_n$ and $h_p = \prod_l h_l = 1$.

We gauge it by relaxing the constraint $h_p = 1$.

As with the continuum presentation of the $\mathbb{Z}_n$ theory above, there are several ways to do it:

- Make the Boltzmann weight an invariant function, e.g. a function of $|\text{Tr } U_p|^2$, or
- Add a $\mathbb{Z}_n$ gauge field $B_p$ on the plaquettes...
From $SU(n)$ to $SU(n)/\mathbb{Z}_n$ on the lattice

- Add a $\mathbb{Z}_n$ gauge field $B_p$ on the plaquettes. In order not to add new dof, add an integer modulo $n$ Lagrange multiplier $\nu_c$ on the cubes. The Boltzmann weight is

$$\left(\prod_c B_c^{\nu_c}\right) \prod_p f(B_p \text{Tr } U_p)$$

More precisely, $V_c = \exp\left(2\pi i \frac{\nu_c}{n}\right) \in \mathbb{Z}_n$ are gauge fields on the dual of the cubes.

The first factor is a $\mathbb{Z}_n$ topological gauge theory.

Can also add the discrete $\theta$-parameter on the lattice...
From $SU(n)$ to $SU(n)/\mathbb{Z}_n$ on the lattice

The $SU(n)$ Wilson loop $\text{Tr}(U_l U_l ... U_l)$ is not invariant under the 1-form symmetry. But

$$W_U = \text{Tr}(U_l U_l ... U_l) B_p B_p ... B_p$$

is gauge invariant – we tile it with a topological surface. It can still detect confinement.

The ‘t Hooft operator is

$$\mathcal{T} = V_c V_c ... V_c$$

with the product over all the cubes penetrated by a curve. The presence of $\mathcal{T}$ changes the constraint on $B_c$ in those cubes.

Closed surface operators $W_B = B_p B_p ... B_p$ measures the ‘t Hooft magnetic flux $w_2$. 
Duality in the $2d$ Ising model

• The dynamical variables are $\mathbb{Z}_n$ spins $S_s$ on the sites and the Boltzmann weight is $\prod_{s, \ell} f(S_s \bar{S}_{s+\ell})$.

• After duality the Boltzmann weight is

$$\left( \prod_{p^*} S_{p^*}^{v_{p^*}} \right) \prod_{\ell^*} \tilde{f}(\bar{S}_{s^*} V_{\ell^*} \bar{S}_{s^*+\ell^*})$$

  – The variables are on the dual lattice.
  – The first factor is a topological $\mathbb{Z}_n$ gauge theory; the $\mathbb{Z}_n$ gauge field $V_{\ell^*}$ is flat.
  – Locally, pick the gauge $V_{\ell^*} = 1$ to find another Ising system.
  – Globally, we need to keep the topological sector.

• The $2d$ Ising model is dual to the $2d$ Ising model coupled to a $\mathbb{Z}_n$ topological gauge theory – an orbifold of Ising.
Duality in $\mathbb{Z}_n$ lattice gauge theory

• It is often stated that the $3d$ $\mathbb{Z}_n$ lattice gauge theory is dual to the Ising model and the $4d$ $\mathbb{Z}_n$ lattice gauge theory is selfdual.

• More precisely, we need to couple them to a $\mathbb{Z}_n$ topological lattice gauge theory:
  – In $3d$ it is an ordinary (0-form) $\mathbb{Z}_n$ gauge theory
  – In $4d$ it is a 1-form $\mathbb{Z}_n$ topological gauge theory – variables on the plaquettes and the cubes are constrained to be 1.

• In $4d$ $\mathbb{Z}_n$ lattice gauge theory
  – At strong coupling – confinement
  – At weak coupling – the ordinary $\mathbb{Z}_n$ gauge symmetry is unbroken – a topological phase, not Higgs.
  – Duality exchanges Higgs and confinement. But since this system is not quite selfdual, there is no contradiction.
Answers to the seemingly unrelated questions

1. 2d Ising is not selfdual. It is dual to an orbifold of Ising – Ising coupled to a topological $\mathbb{Z}_n$ gauge theory.

2. The spacelike commutation relations

\[ W(\gamma) \mathcal{T}(\gamma') = e^{2\pi i \ell(\gamma, \gamma')/n} \mathcal{T}(\gamma') W(\gamma) \]

are consistent because at least one of these operators includes a (topological) surface.

3. The global $\mathbb{Z}_n$ symmetry of a 4d $SU(n)$ gauge theory on a circle originates from a global 1-form symmetry in 4d. Gauging it has the effect of changing the 4d gauge group to $SU(n)/\mathbb{Z}_n$. 

Answers to the seemingly unrelated questions

4. Not all consistency conditions on $c_k$ in $\mathcal{Z} = \sum c_k \mathcal{Z}_k$ are understood. For a given gauge group there are several distinct consistent options, including changing the sum over instanton sectors. They can be described by coupling the system to a TQFT.

5. Depending on $p$ there are two (actually, 4) distinct $SO(3)$ gauge theories in $4d$, $SO(3)_\pm$ [Gaiotto, Moore, Neitzke].

In $\mathcal{N} = 4$ SUSY [Aharony, NS, Tachikawa]

\[
\begin{array}{ccc}
SU(2) & \leftrightarrow & SO(3) \\
\cup & S & \leftrightarrow & T \\
T & \cup & SO(3)_- \quad S
\end{array}
\]

which is consistent with $(ST)^3 = 1$. 
Answers to the seemingly unrelated questions

6. Unified descriptions of orbifolds and 4d gauge theories
   • In **orbifolds** we start with a system with a global symmetry.
     – Background gauge field – twisted boundary conditions
     – Gauging the symmetry by summing over these sectors
     – This removes operators and includes others
     – Discrete torsion: different coefficients for the sectors
   • In **4d gauge theories** the global symmetry is a 1-form symmetry
     – Twisted sectors are bundles of a quotient of the gauge group
     – Gauging the 1-form global symmetry – summing over sectors
     – This changes the line and surface operators
     – Discrete $\theta$-parameter – different coefficients for the sectors
Conclusions

It is interesting to couple an ordinary QFT to a TQFT.
• The resulting theory can have a different local structure.
• More generally, it has different line and surface operators.
• When placed on manifolds other than \( \mathbb{R}^D \) the effects are often more dramatic.
• Such a coupling to a TQFT can describe the difference between a theory with gauge group \( G \) and a theory with gauge group \( G/C \), e.g. \( SU(n) \) and \( SU(n)/\mathbb{Z}_n \).
• It also allows us to describe additional coupling constants like discrete \( \theta \)-parameters, or restrictions on the range of the ordinary \( \theta \)-angle in a manifestly local way.
• Such added TQFT also resolve problems with duality (2d Ising, 4d lattice gauge theories, \( \mathcal{N} = 1, 4 \) theories in 3d and 4d).
Thank you for your attention