Exam in Statistical Physics

May 2008

1 Landau theory and domain walls

The free energy density f of the order-parameter $\phi(x)$ is :

$$f = \frac{1}{2}g(\phi')^2 + \frac{1}{2}a\phi^2 - \frac{1}{4}\phi^4 + \frac{1}{6}\phi^6.$$

a) Consider the homogeneous system, $\phi' = 0$: Examine the five extrema of $f(\phi)$ as a function of a to locate the stable and the metastable phases. Draw the shape of $f(\phi)$ in each regime of a. What happens at $a = 1/4$? at $a = 3/16$? at $a = 0$?

b) The free-energy is the functional is $F = \int f(x)dx$. Take the variational derivative δF (Euler-Lagrange equation) to show that $\phi(x)$ obeys $g\phi'' = a\phi - \phi^3 + \phi^5$. Show that $g(\phi')^2/2 = a\phi^2/2 - \phi^4/4 + \phi^6/6 - f_0$, where the constant f_0, the free energy where $\phi' = 0$, is determined by boundary conditions (Hint: Newton's law).

c) At $a = 3/16$, consider a system where a disordered phase, $\phi = 0$, coexists with an ordered phase, $\phi_0 > 0$ (ϕ_0 was found in (a)) and the boundary conditions are therefore $\phi(-\infty) = 0$, $\phi(+\infty) = \phi_0$. Show that, in this case, $g(\phi')^2/2 = \phi^2(\phi_0^2 - \phi^2)^2/6$, and find the profile $\phi(x)$ that describes this domain wall (a useful integral is $\int [z(z_0^2 - z^2)]^{-1}dz = -(2z_0^2)^{-1}\ln(z_0^2/z^2 - 1)$). What is the length scale of $\phi(x)$?

d) The domain wall energy is $E = \int_{-\infty}^{+\infty} dx(f(x) - f_0)$. Show that $E = \int_{-\infty}^{+\infty} dx(f(x) - f_0) = g\int_{-\infty}^{+\infty} dx(\phi')^2 = g\int_{\phi(-\infty)}^{\phi(+\infty)} d\phi(\phi')$. Calculate E explicitly for $a = 3/16$.

2 Thermal chord

A chord of length l_0 and mass per unit length μ is fixed at both ends and tightened to a tension τ . The chord is in equilibrium with a thermal bath of temperature T. Consider the transverse displacement of the chord $y(x,t)$, where x is the coordinate along the chord.

a) Show that, for small displacements, the equation that governs the motion of the chord (Newton's law) is

$$\frac{\partial^2 y}{\partial t^2} = \frac{\tau}{\mu}\frac{\partial^2 y}{\partial x^2},$$

and the total energy of the chord is

$$\mathcal{H} = \frac{1}{2}\int_0^{l_0} dx \left[\mu\left(\frac{\partial y}{\partial t}\right)^2 + \tau\left(\frac{\partial y}{\partial x}\right)^2\right]$$

(Hint: The elastic energy can be written as $\tau(l - l_0)$, where $l = \int dl$ is the total length of the chord and l_0 the projected length).

b) (i) The boundary conditions are $y(0) = y(l_0) = 0$. Find the normal modes of the chord, i.e. their discrete spectrum of wave-vectors k_n, their frequencies ω_n and the dispersion relation.

(ii) Express $\mathcal{H}$ in terms of the amplitudes of these modes A_n. Write the partition function $Z(A_n)$. What is the ensemble average $\langle A_n^2 \rangle$? Which principle have you used?

c) Consider the mean square displacement at the middle of the chord $y_0^2 = \langle y^2(l/2)\rangle$. (i) Which modes contribute to y_0^2? Express y_0^2 as a function of these modes. (ii) Finally, find y_0^2 as a function of the system parameters (τ, μ, T, l_0). A useful series is $\sum_{m=0}^{\infty}(2m+1)^{-2} = \pi^2/8$.

3 Squeezing and stretching a polymer

In a simplified view, a 2D polymer is a random walk on an infinite square lattice, where the lattice constant is the length of a monomer a. A polymer starts at the origin 0 and at each step can proceed towards any of the $q = 4$ neighboring sites (we allow it to cross itself).

a) Consider a polymer of length N (i) Find the overall number of configurations that start at 0 and end anywhere, T_N. (ii) Show that the average square end-to-end distance of the polymer R_0^2 is $R_0^2 = Na^2$. (iii) Denote by $\Gamma_N(\mathbf{r})$ the number of polymer configurations that start at 0 and end at $\mathbf{r}$ ($T_N = \sum_{\mathbf{r}} \Gamma_N(\mathbf{r})$). Find the distribution function $p(\mathbf{r}) = \Gamma_N(\mathbf{r})/T_N$ and show that for large N it becomes Gaussian (you may neglect prefactors),

$$p(\mathbf{r}) \sim \frac{1}{N}\exp\left(-\frac{r^2}{Na^2}\right).$$

(Hint: the random walks in the x and the y directions are independent).

b) Find the entropy $S(\mathbf{r}) = \ln \Gamma_N(\mathbf{r})$ (Use S_0 to denote the $\mathbf{r}$-independent part of the entropy). Find the free energy $F(\mathbf{r}) = -TS(\mathbf{r})$ (Neglect the $\mathbf{r}$-independent energy).

c) (i) Apply forces $\mathbf{f}$ and $-\mathbf{f}$ at the ends of the polymer and use the relation $\mathbf{f} = \partial F/\partial \mathbf{r}$ to find the average elongation as a function of the force $r_f = \langle r \rangle$.

(ii) What is the "spring constant" of the polymer?

(iii) One applies a constant force on a polymer while heating it. What happens to the polymer? What happens at $T \to 0$?

d) Now the same polymer is squeezed between two parallel walls at a distance D from each other, $a \ll D \ll R_0$.

(i) What is the expected end-to-end distance parallel to the walls? (Hint: x and y random walks are independent). Show that, as a result, the only dimensional parameters in the problem are R_0, D and T. Squeezing reduces the entropy of the polymer by ΔS (ΔS is dimensionless).

(ii) Show from scaling arguments that $\Delta S \sim (R_0/D)^\alpha$ with an exponent α, which cannot be found from scaling.

(iii) To find α, consider the scaling of ΔS in N, $\Delta S \sim N^\beta$. What must be therefore β? and α?

4 Maximum entropy principle

Consider two systems A and B with Shannon entropies $S_A = -\sum_a p_a \ln p_a$ and $S_B = -\sum_b p_b \ln p_b$, where a and b are the corresponding states. The states of the composite system AB are labeled by ab and the corresponding entropy is $S_{AB} = -\sum_{a,b} p_{ab} \ln p_{ab}$.

a) (i) If A and B are weakly interacting $p_{ab} = p_a p_b$. Show that in this case the entropy is additive, $S^0_{AB} = S_A + S_B$.

(ii) Now A and B are strongly interacting and at least for some states, $p_{ab} \neq p_a p_b$. Show that

$$\Delta S = S_{AB} - S^0_{AB} = S_{AB} - (S_A + S_B) = \sum_{a,b} p_{ab} \ln \frac{p_a p_b}{p_{ab}}. \quad (1)$$

(iii) What is the sign of ΔS (Hint: $\ln x \leq x - 1$). Show that $\Delta S = 0$ if and only if $p_{ab} = p_a p_b$ for every a and b . Discuss the physical meaning of the result.

One may think of certain networks, such as the internet, as a directed graph G of N vertices. Every pair of vertices, say i and j, can be connected by multiple edges (e.g. hyperlinks) and loops may connect edges to themselves. The graph can therefore be described in terms of an adjacency matrix A_{ij} whose N^2 elements can be any non-negative integer, 0, 1, 2...

b) The entropy of the ensemble of all possible graphs is $S = -\sum_G p_G \ln p_G = -\sum_{\{A_{ij}\}} p(A_{ij}) \ln p(A_{ij})$. Consider such an ensemble with an average number of edges per vertex $\langle k \rangle$.

(i) Write an expression for the number of edges per vertex k for a certain graph A_{ij}. Use the maximum entropy principle to calculate $p_G(A_{ij})$ and the partition function Z (denote the Lagrange multiplier by τ). What is the equivalent of the Hamiltonian? What are the degrees of freedom? What kind of "particles" are they? Are they interacting?

(ii) Calculate the free energy $F = -\ln Z$, and express it in terms of τ. Is it extensive? If it is in what number?

(iii) Write down an expression for the occupation number $\langle A_{ij} \rangle$ as a function of τ. What is the name of this statistics? What is the "chemical potential" and why?

(iv) Express F as a function of τ and as a function $\langle k \rangle$. Express p_G as a function of k, $\langle k \rangle$ and N.