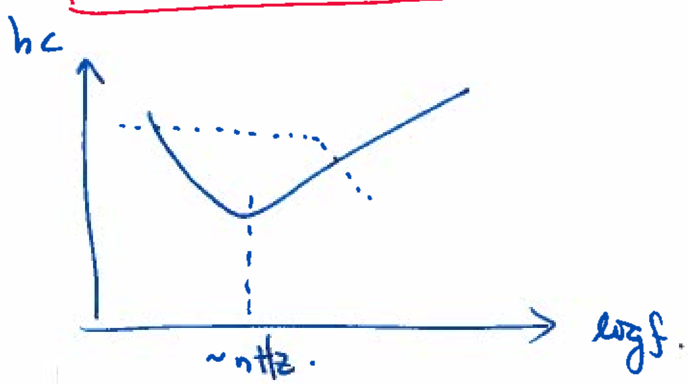
Brief Intro to NANOGrav Paper [2009.04496].

For more pedagogical purposes; introduce

1) Characteristic strain

$$\boxed{h_c \equiv A_{GWB}\left(\frac{f}{f_{1yr}}\right)^{\alpha},} \qquad f_{1yr} \equiv \frac{1}{1yr} = 3.17\times10^{-8}\ \mathrm{Hz}.$$



α : describes the nature of GW source

→ for SMBBH, $\alpha = -2/3$

A_{GWB} : describes the sum over the stochastic GW background produced by such a source.

I will: ~~Der~~ introduce h_c

Derive h_c for a general stochastic background

Derive $\alpha = -2/3$ scaling for SMBBH

2) Timing-Residual Cross-power spectral density.

$$\langle R_a(t) R_b(t) \rangle \equiv \int_0^\infty df \, S_{ab}(f).$$

$$S_{ab} = \left[\tfrac{3}{2} C(\theta_{ab}) \right] \times \frac{h_c^2}{12\pi^2 f^3}.$$

only Hellings Down curve would tell us the gravitational wave nature of the sources.

$$\boxed{S_{ab} \equiv \Gamma_{ab} \frac{A_{GWB}^2}{12\pi^2} \left(\frac{f}{f_{1yr}} \right)^{2\alpha-3} f_{1yr}^{-3}}$$

Show Fig 1.

Show Fig 5.

Stochastic GW Background from SMBBH.

f_s, E_s: frequency and GW energy in the source frame

f, E: " " " detector frame

θ^i: parameters of binary (mass, spin, etc)

$$f = \frac{f_s}{1+z}, \quad E = \frac{E_s}{1+z}, \quad z: \text{redshift.}$$

Let $n(z, \theta^i)$ be the no. density of sources in the parameter ranges $[\theta^i, \theta^i + d\theta^i]$ and redshift range $[z, z+dz]$, the total energy density at a frequency interval $[\log f, \log f + d\log f]$ is

$$\frac{d\rho_{gw}}{d\log f} = \int dz\, d\theta^i\, n(z, \theta^i)\, \frac{dE(f_s)}{d\log f_s}$$

$$= \int \frac{dz}{1+z}\, d\theta^i\, n(z, \theta^i) \left[\frac{dE_s}{d\log f_s}\right]_{f_s = (1+z) f} \quad \text{—— (1)}$$

Definition of characteristic strain:

$$\langle h_{ij} h_{ij} \rangle := 2 \int d\log f \; h_c^2(f)$$

$$\rho_{gw} = \frac{1}{32\pi G} \langle \dot{h}_{ij} \dot{h}_{ij} \rangle \xrightarrow{FT} \frac{(2\pi f)^2}{32\pi G} \langle h_{ij} h_{ij} \rangle$$

$$\frac{d\rho_{gw}(f)}{d\log f} = \frac{(2\pi f)^2}{32\pi G} \frac{d\langle h_{ij} h_{ij} \rangle}{d\log f}$$

$$= \frac{\pi}{4G} f^2 h_c^2(f) \quad — ②$$

Sub. ② into ①,

$$\boxed{h_c^2(f) = \frac{4G}{\pi f^2} \int \frac{dz}{1+z} \, d\theta_i \; n(z,\theta_i) \left[\frac{dE_s}{d\log f_s} \right]_{f_s = (1+z)f}} \quad — ③$$

Phinney 2001

→ Amplitude contains information of the sum of no. density, parameters of binary, over the cosmic volume.
(Astrophysical population properties)

→ f-scaling: nature of the source.

α-Scaling for SMBBH

$$E_{orbit} = E_K + E_P$$

$$= \frac{1}{2}\nu M_{tot} v^2 - \frac{G m_1 m_2}{R}, \quad \nu \equiv \frac{m_1 m_2}{M_{tot}^2}$$

Virial Theorem, $v^2 = \frac{G M_{tot}}{R}$

$$\therefore E_{orbit} = -\frac{G m_1 m_2}{2R}$$

From Kepler's Law, $G M_{tot} = \Omega^2 R^3$

$$= (\pi f)^2 R^3$$

$$R^{-1} = \frac{(\pi f)^{2/3}}{(G M_{tot})^{1/3}}.$$

$$\therefore E_{orbit} \propto \frac{m_1 m_2}{M_{tot}^{1/3}} f^{2/3} \propto M_c^{5/3} f^{2/3}$$

$$E_{gw} \sim E_{orbit} \sim M_c^{5/3} f^{2/3}$$

$$\frac{dE_{gw}}{d\log f} = f \frac{dE_{gw}}{df} \propto M_c^{5/3} f^{2/3}$$

$$h_c^2(f) \propto f^{-2+2/3} \propto f^{-4/3}$$

$$\therefore h_c(f) \propto f^{-2/3}$$

Short Intro of Final Parsec Problem.

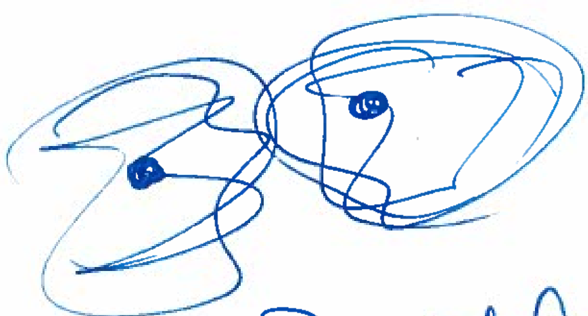
Kepler's law: $GM_{tot} = \Omega^2 R^3 = (\pi f)^2 R^3$

$$R \sim (0.1pc)\left(\frac{M_{tot}}{10^{10}M_\odot}\right)^{1/3}\left(\frac{10^{-9}Hz}{f}\right)^{2/3}$$

For ~~[illegible]~~PTAs to observe these SMBH inspirals, the binary separation has to be rather small

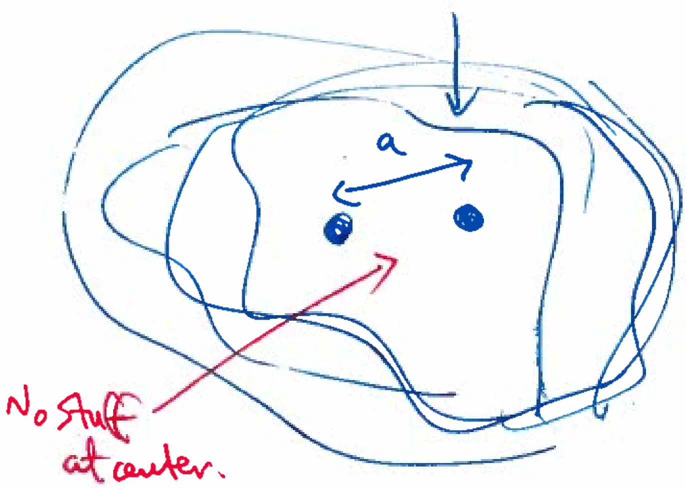
BUT ~~[illegible]~~ ~~[illegible]~~ theoretical have two SMBH come this close. it turns out it is hard to

~~[illegible]~~



Loss of kinetic energy of a body when it interacts gravitationally with ~~the~~ a surrounding medium.

→ Dynamical friction dominates the 2-body dynamics in the early stage (when there is a relative motion between medium & object)



$$a \simeq 3.5pc \left(\frac{180\,km\,s^{-1}}{\sigma_c}\right)^2 \left(\frac{M}{10^8 M_\odot}\right)$$

[Until BBH orbital velocity is ~ velocity dispersion of stars]

τ_{GW}(to merge)

$$\simeq 5.8 \times 10^{14} \text{yr} \left(\frac{a}{1\text{pc}}\right)^4 \left(\frac{10^8 M_\odot}{M_1}\right)^3 \frac{M_1^2}{M_2(M_1+M_2)}$$

$\Downarrow$

Longer than the age of the universe!

Resolution?

1) MANY uncertainties in the estimates above

2) Stellar dynamics can harden the two-bodies

Dynamical Friction

Chandrasekhar's formula

$$-\frac{dv_M}{dt} = -16\pi^2 (\ln\Lambda)\, G^2 m_1 (M_1+m_2) \frac{1}{v_M^3} \int_0^{v_M} v^2 f(v)\, dv \; \underline{v_M}$$

"$M-\sigma$" relation

$$\log_{10} \frac{M_{BH}}{M_\odot} = \alpha + \beta \log_{10} \frac{\sigma}{\sigma_0}$$

(Relationship between supermassive BH at the centre of galaxies & the velocity dispersion σ of stars in the inner region of the galaxy).