Introduction to Pulsar Timing Arrays

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1 Introduction

The pulses of radio emission from a pulsar are very regular; the typical secular changes in period are $\sim 10\ \mu$s per year. This means that earth and a pulsar have the potential to be a gravitational wave detection system like that depicted schematically in Figure 1.

Figure 1: Pulsar, earth, and a gravitational wave

If an array of pulsars is observed every t amount of time (typically about weekly) over a total time T (about 10 years), the range of frequencies that can be detected is

$$\frac{1}{T} \lesssim f \lesssim \frac{1}{t}, \tag{1}$$

which translates to

$$3 \times 10^{-9}\, Hz \lesssim f \lesssim 1 \times 10^{-6}\, Hz, \tag{2}$$

a noticeably different frequency range from LIGO's ten to a few thousand Hz.

Of course, there are several factors that can affect radio emission arrival time beyond gravitational waves. This includes intrinsic variability in the pulsar and the motion of the observer in the solar system. These effects must be carefully removed if possible, and significant noise effects may still remain. Often, an array of pulsars is used to extract a signal from the noise: with an array of pulsars, the effect of a gravitational wave background (which itself is a form of noise for a stochastic background) is correlated across the sources, but other noise sources are not.

2 Gravitational wave effect on one pulsar

We work with the positions of the earth and the pulsar fixed (with a distance d_a between them) and set $c = 1$. The most general setup is shown in Figure 2. The GW is propagating in the $\hat{z}$ direction, with an amplitude of the form
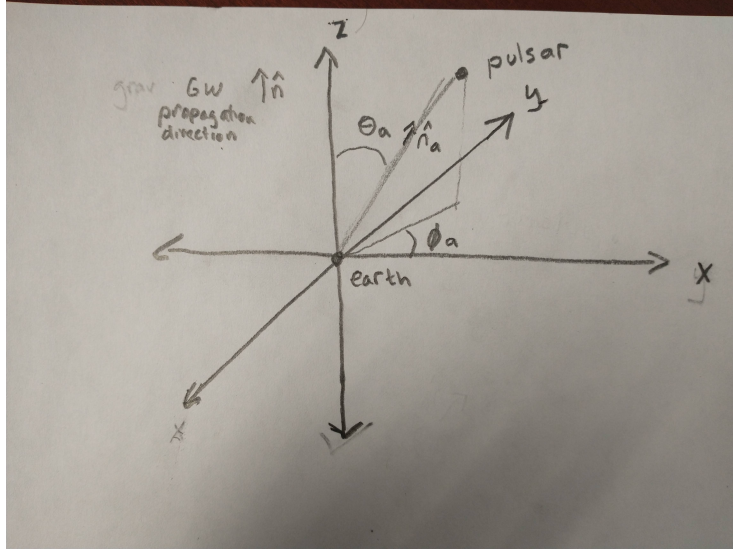

Figure 2: The setup.

$$h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

The pulsar emits one pulse at t_{em} and another at time $t'_{em} = t_{em} + T_a$

Let's start by considering a simplified case where $\hat{n}_a$ is in the $\hat{x}$ direction. In that case, the path of photons traveling from the pulsar to the observer is given by

$$ds^2 = -dt^2 + [1 + h_+ (t, x)] \, dx^2 = 0. \tag{4}$$

Expansion gives

$$dt = \left[1 + \frac{1}{2} h_+ (t, x)\right] dt. \tag{5}$$

Then, we find that the following relation

$$t_{obs} - t_{em} = d_a + \frac{1}{2} \int_{t_{em}}^{t_{obs} \approx t_{em} + d_a} dt' h_+ \left[t', (t_{em} + d_a - t')\right]. \tag{6}$$

For the next pulse, we have

$$t'_{obs} = t_{em} + T_a + d_a + \frac{1}{2} \int_{t_{em}+T_a}^{t_{em}+T_a+d_a} dt' h_+ \left[t', (t_{em} + T_a + d_a - t')\right]$$

$$= t_{em} + T_a + d_a + \frac{1}{2} \int_{t_{em}}^{t_{em}+d_a} dt' h_+ \left[t' + T_a, (t_{em} + d_a - t')\right] \quad (7)$$

We wish to find the quantity ΔT_a defined by

$$t'_{obs} - t_{obs} = T_a + \Delta T_a. \quad (8)$$

This is

$$\Delta T_a = \frac{1}{2} \int_{t_{em}}^{t_{em}+da} dt' \left\{ h_+ \left[t' + T_a, x\left(t'\right)\right] - h_+ \left[t', x\left(t'\right)\right] \right\}. \quad (9)$$

We define the quantity

$$z_a = \frac{\Delta T_a}{T_a} \quad (10)$$

expanding the integrand in (9) to lowest order in T_a and integrating gives

$$z_a\left(t_{obs}\right) = \frac{1}{2} \left\{ h_+\left(t_{obs}, 0\right) - h_+\left(t_{em}, d_a\right) \right\} \quad (11)$$

(Here, the observer's position is at 0 and the pulsar's is at d_a.) With the full geometry of in Figure 2, this becomes

$$z_a\left(t_{obs}\right) = \frac{n_a^i n_a^j}{2\left(1 + \hat{n} \cdot \hat{n}_a\right)} \left\{ h_{ij}\left(t_{obs}, 0\right) - h_{ij}\left[t_{em}, d_a \hat{n}_a\right] \right\} \quad (12)$$

3 Hellings-Downs Curve

Space is filled with gravitational waves of all frequencies coming from all directions. This is described by

$$h_{ij} = \sum_{A=+,\times} \int_{\infty}^{\infty} df \int d^2\hat{n} \, \tilde{h}_A\left(f, \hat{n}\right) e_{ij}^A\left(\hat{n}\right) e^{-i2\pi f(t - \hat{n}\cdot\vec{x})}. \quad (13)$$

Asuming that the GW background is stationary, isotropic, and unpolarized, it can be shown via a non-trivial integration that

$$\langle z_a\left(t\right) z_b\left(t\right) \rangle \propto F\left(\theta_{ab}\right) = \left(\frac{1 - \cos\theta_{ab}}{2}\right) \log\left(\frac{1 - \cos\theta_{ab}}{2}\right) - \left(\frac{1 - \cos\theta_{ab}}{12}\right) + \frac{1}{3},$$

$$(14)$$

with θ_{ab} the angle of separation between pulsars a and b. This curve is shown in Figure 3.

4 Futher Reading

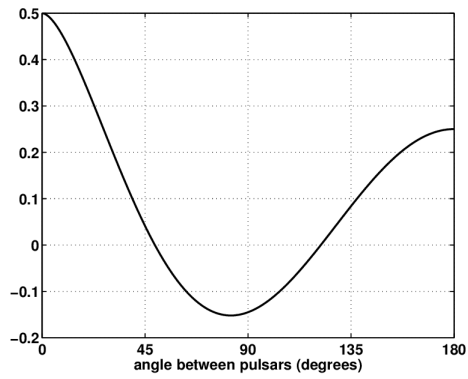These notes are primarily based on Maggiore's *Gravitational Waves*.

Figure 3: Hellings-Downs Curve, from Romano, Joseph D., and Neil J. Cornish. "Detection methods for stochastic gravitational-wave backgrounds: a unified treatment." Living reviews in relativity 20.1 (2017): 2.