*) Detectors

-) size of the displacement?
-) Interferometer → basic concept
-) How come it can be measured.
sources of noise.

*) Basic properties of the signal

- $A(f)$
- $\phi(f)$

- Time frequency diagram.
- Duration of signal
- Chirp mass

⊛ Searches

A) Ideal search

-) Matched filter: why?

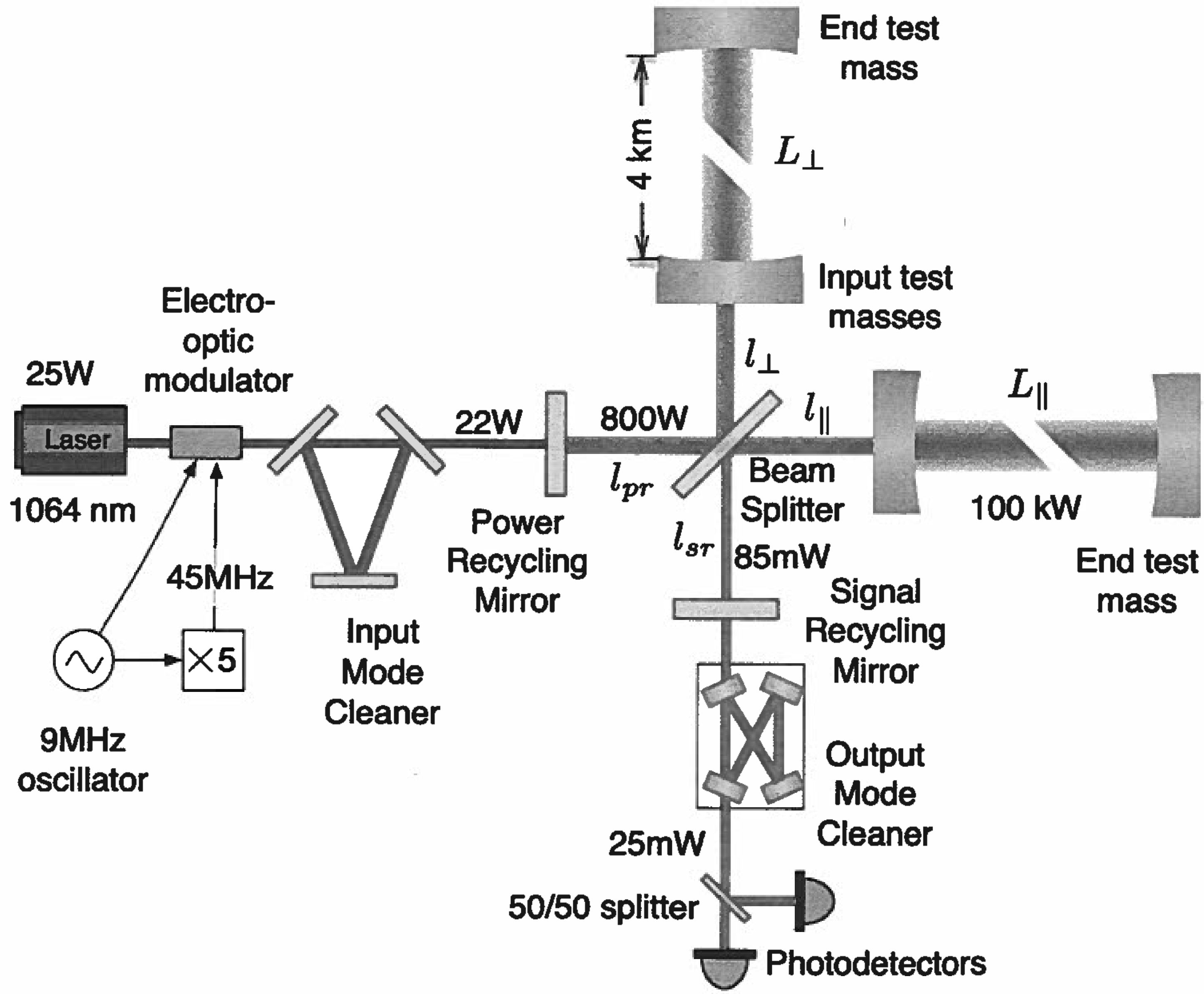
-) Template banks

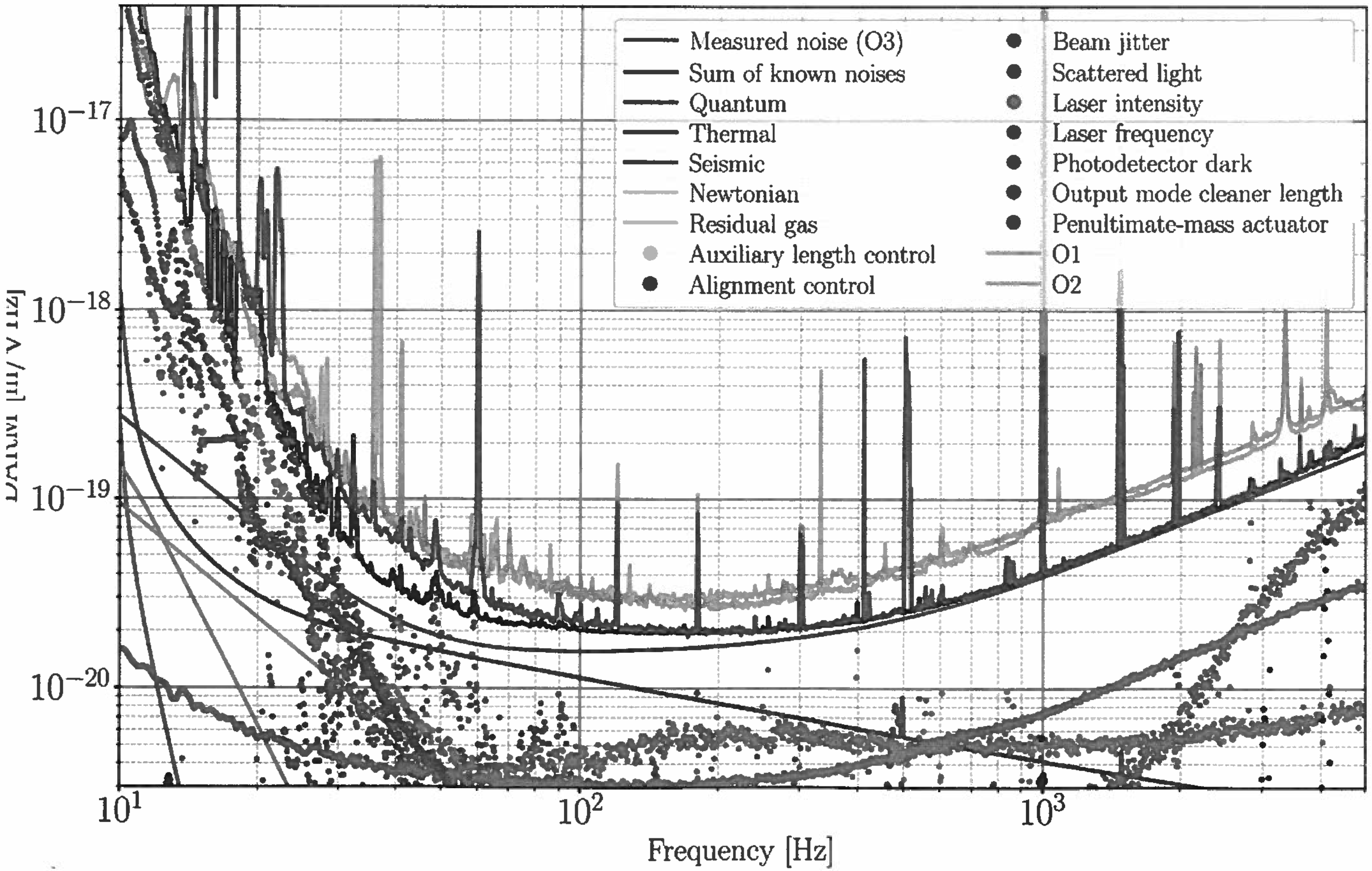
-) Simple example with chirp mass

-) look elsewhere effect → order of magnitude of bar

B) Non-ideal issues.

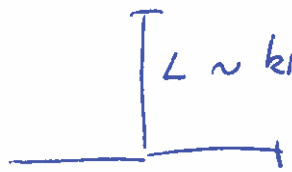
-) list them


End test mass
4 km
$L_\perp$
Input test masses
$l_\perp$
$l_\parallel$
$L_\parallel$
25W
Electro-optic modulator
Laser
1064 nm
45MHz
×5
9MHz oscillator
22W
800W
Input Mode Cleaner
Power Recycling Mirror
l_{pr}
Beam Splitter
l_{sr}
85mW
100 kW
End test mass
Signal Recycling Mirror
Output Mode Cleaner
25mW
50/50 splitter
Photodetectors


DARM [m/√Hz]
10^−17
10^−18
10^−19
10^−20
10^1
10^2
10^3
Frequency [Hz]
Measured noise (O3)
Sum of known noises
Quantum
Thermal
Seismic
Newtonian
Residual gas
Auxiliary length control
Alignment control
Beam jitter
Scattered light
Laser intensity
Laser frequency
Photodetector dark
Output mode cleaner length
Penultimate-mass actuator
O1
O2

LIGO

GW detection



$h \sim 10^{-21}$

150914

$\Delta L \sim km \times 10^{-21}$

$\sim 10^{-18}\ m \sim 10^{-3}$ femtometers

Fermi

1 proton $\sim$ 1 fm

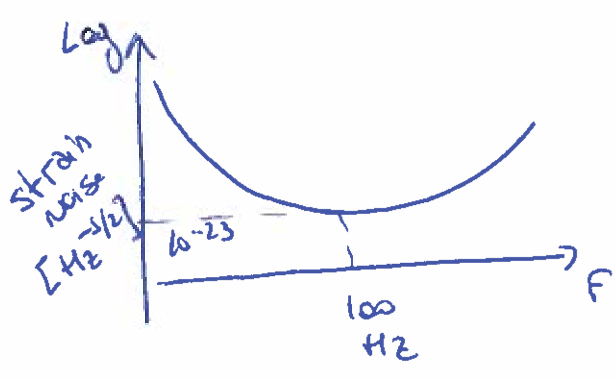
How is this possible?

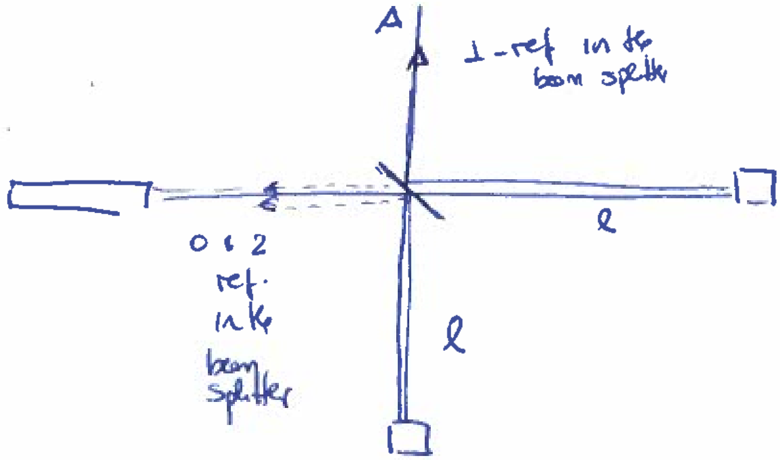
$f_{GW} \sim 100\ Hz$

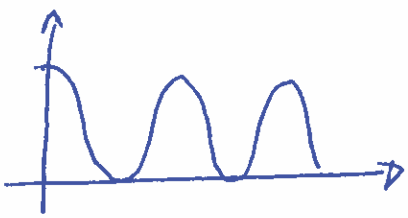
$$\lambda_{GW} = c/f_{GW} \sim \frac{3\ 10^5\ km/s}{100 Hz} \sim 3 \times 10^3\ km \sim 10^3\ L_{LIGO}$$

$\lambda_{Laser} \sim 1\ \mu m \sim 10^{-6}\ m$

$$\frac{\Delta L}{\lambda} \sim 10^{-12}$$







$$\dot N = \frac{I}{h\nu}(1-\cos\phi)$$ = # of photons in the output per unit time

$$\phi = \frac{2\pi}{\lambda}\, 2(l_1 - l_2)$$

$$\boxed{\phi = \frac{4\pi}{\lambda}\Delta l}$$

$$\boxed{\delta \dot N = \frac{I}{h\nu}\sin\phi\,\frac{4\pi}{\lambda}\,\delta l}$$

$$\delta l = \frac{h\nu}{I\sin\phi}\,\frac{\lambda}{4\pi}\,\delta\dot N$$

$$N = \dot N T \pm \sqrt{N}$$ Poisson noise

$$(\delta N)^2 = N$$

$$(\delta\dot N)^2 = \left(\frac{\delta N}{T}\right)^2 = \frac{N}{T^2} = \frac{\dot N}{T}$$

$$\Delta f = \frac{1}{T}$$

$$(\delta l)^2 = \left(\frac{h\nu\lambda}{I\sin\phi\, 4\pi}\right)^2 \delta\dot N$$

$$(\delta \ell)^2 = \left(\frac{h\nu}{I}\,\frac{\lambda}{4\pi}\right)^2 \frac{1}{(1-\cos^2\phi)}\,\frac{I}{h\nu}\,(1-\cos\phi)\,\Delta f$$

$$\propto \frac{h\nu}{I}\,\frac{\lambda^2}{(1+\cos\phi)}\,\Delta f \;\propto\; \frac{h\nu\lambda^2}{I\cos(\phi/2)}\,\Delta f$$

$$\boxed{(\delta\ell)^2\Big|_{\text{shot noise}} \propto \frac{hc\lambda}{I\cos(\phi/2)}\,\Delta f}$$

Minimum when $\phi = 0$ $\Rightarrow$ DARK PORT

$\propto 1/I$

$$(\delta\ell)^2\Big|_{SN} \propto \frac{\lambda_{laser}}{I_0}$$

<u>Radiation pressure fluctuations</u>

m

$$\Delta p = \frac{2h\nu}{c}$$

$$m\ddot{x} = F$$

$$x = \frac{F}{m\omega^2}$$

$$\left(\frac{\delta p}{T}\right)^2 = \left(\frac{2h\nu}{c}\right)^2 \frac{N}{T^2}$$

$$\left(\frac{\delta p}{T}\right)^2 = (\delta F)^2 = \left(\frac{2h\nu}{c}\right)^2 \left(\frac{I}{h\nu}\right) \Delta F$$

$$(\delta x)^2 = \frac{1}{m^2\omega^4}\left(\frac{2h\nu}{c}\right)^2\left(\frac{I}{h\nu}\right)\Delta F$$

$$\boxed{(\delta\ell)^2\Big|_{\text{rad pressure}} \propto \frac{I_0 h\nu}{m^2\omega^4 c^2}\,\Delta f} \qquad \boxed{\propto \frac{I_0}{\lambda_{laser} m^2 \omega^2}}$$

③

$$(\delta e)^2 = (\delta e)^2_{SN} + (\delta e)^2_{\substack{\text{rad}\\\text{pressure}}} \sim \left(\underbrace{\frac{hc\lambda}{I_0}}_{\downarrow \ \frac{hc^2}{I_0 \nu}} + \frac{I_0 h\nu}{m^2 \omega^4 c^2} \right) \Delta F$$

There is a best intensity

$$I_0 \propto \frac{m\omega^2 c^2}{\nu}$$

$$I_0^2 \propto \cancel{h} c\lambda \frac{m^2\omega^4 c^2}{\cancel{h}\nu} \propto m^2 \omega^4 \frac{c^4}{\nu^2}$$

$$\boxed{(\delta e)^2\Big|_{min}} \sim \frac{\Delta f \cancel{\nu}}{m\omega^2 \cancel{c^2}} \frac{h\cancel{c^2}}{\cancel{\nu}} \sim \boxed{\frac{4\hbar}{m\omega^2} \Delta F}$$

strain $L \sim \delta e$

$\omega \sim$ ~~100 Hz~~ 10s of Hz

$m \sim 1$ kg $\qquad \hbar$

$L \sim$ km

$$(\text{strain})^2 \sim \frac{\hbar}{m(L\omega)^2} \Delta F$$

$$(L\omega) \sim 10^4 \ (m/s)^2 \qquad m(L\omega)^2 \sim 10^7 \text{ J}$$

$$\hbar \sim 10^{-34} \text{ Js}$$

$$(\text{strain})^2 \sim (10^{-34} \times 10^{-7}) \text{ s} \ \Delta f \sim 10^{-41} \text{ Hz}^{-1} \ \Delta f$$

Uncertainty principle

$$(\Delta p)^2 \sim \frac{\hbar^2}{(\Delta x)^2}$$

$$(\Delta F)^2 \sim \frac{\hbar^2}{(\Delta x)^2}\,\frac{1}{T^2}$$

$$\Delta x^2 \sim (\Delta x)^2 + \frac{\hbar^2}{(\Delta x)^2\, m^2 \omega^4}\,(\Delta F)^2$$

$$\Rightarrow \boxed{\Delta x^2\Big|_{min} \sim \frac{\hbar}{m\omega^2}\,\Delta F}$$

$$h^2 = h_s^2 + h_p^2 = \frac{\hbar}{m\omega^2}\left[K + \frac{1}{K}\right]\frac{1}{2}$$

$$K = \frac{4 m \omega^2 c^2}{I_0\, \omega_{laser}}$$

Noise is quantum mechanical in origin

Other sources of noise:

- Thermal noise
- Seismic noise
- Newtonian Noise.

$$h^2\big|_{shot} \propto \frac{h c \lambda}{L^2 I} \propto \frac{h c^2}{I \nu L^2}$$

$$h^2\big|_{pressure} \propto \frac{I_0 \nu \hbar}{m^2 f^4 c^2 L^2} \qquad f = 2\pi \Omega$$

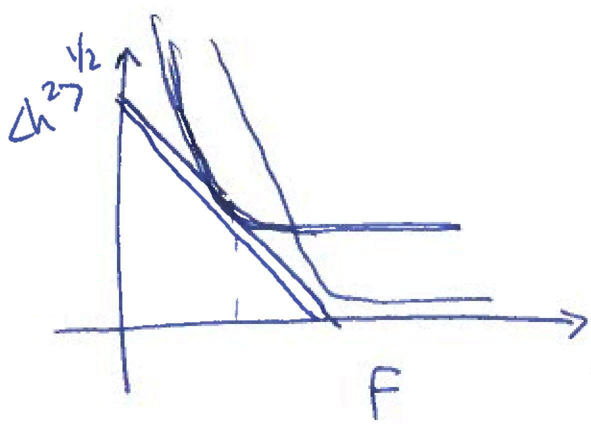
$$h_T^2 = \frac{\hbar}{m f^2 L^2} \left[\frac{m f^2}{I \nu} + \frac{I \nu}{m f^2} \right]$$

$$= \frac{4\hbar}{m \Omega^2 L^2} \, \frac{1}{2} \left(K + \frac{1}{K} \right) \qquad K = \frac{4 P \omega_{Laser}}{c^2 m \Omega^2}$$

$$h_Q^2 = \frac{4 \times 10^{-34}}{40\,\text{kg}\,(10\,\text{Hz})^2\, 16\, 10^8}$$

$$\sim 10^{-34}\, 10^{-1}\, 10^{-2}\, 10^{-7} \sim 10^{-44} \; \cancel{\sim 10}$$

$$\Rightarrow h \sim 10^{-22}\ \text{Hz}^{-1}$$



Thermal Fluctuations

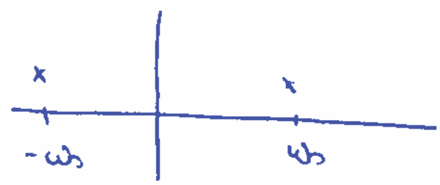
$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F}{m}$$

$$\left((\omega_0^2 - \omega^2) + i\gamma\omega\right) X(\omega) = \frac{F(\omega)}{m}$$

$$X(\omega) = \frac{1}{(\omega_0^2 - \omega^2) + i\gamma\omega} \, \frac{F(\omega)}{m}$$

$\omega_\pm$: $\quad \omega^2 - i\gamma\omega - \omega_0^2 = 0$

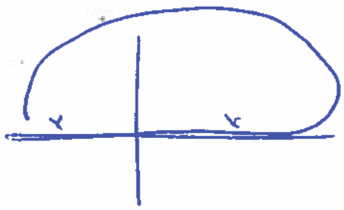
$$\omega_\pm = \frac{i\gamma \pm \sqrt{4\omega_0^2 - \gamma^2}}{2}$$



$$X(\omega) = -\frac{1}{(\omega - \omega_+)(\omega - \omega_-)} \, \frac{F(\omega)}{m}$$

$$|X(\omega)|^2 = \frac{1}{(\omega - \omega_+)(\omega - \omega_+^*)(\omega - \omega_-)(\omega - \omega_-^*)} \, \frac{|F(\omega)|^2}{m^2}$$

$$\int |X(\omega)|^2 \, d\omega = \frac{kT}{\omega_0^2 m}$$ Equipartition



$$\int |x(\omega)|^2 = \frac{|F|^2}{m^2} \int \frac{1}{(\omega-\omega_+)(\omega-\omega_+^*)(\omega-\omega_-)(\omega-\omega_-^*)}$$

$$= \frac{|F|^2}{m^2}\, 2\pi i \left[\frac{1}{(\omega_+-\omega_+^*)} \frac{1}{(\omega_+-\omega_-)(\omega_+-\omega_-^*)} + \frac{1}{(\omega_- - \omega_+^*)} \frac{1}{(\omega_- - \omega_+)} \frac{1}{(\omega_- - \omega_-^*)} \right]$$

$$\omega_+ - \omega_+^* = i\gamma$$
$$\omega_- - \omega_-^* = i\gamma$$

$$(\omega_+ - \omega_-) = (4\omega_0^2-\gamma^2)^{1/2} \qquad (\omega_- - \omega_+) = -(4\omega_0^2-\gamma^2)^{1/2}$$

$$(\omega_+ - \omega_-^*) = \underbrace{(4\omega_0^2-\gamma^2)^{1/2}}_{A} \underbrace{+ i\gamma}_{+iB} \qquad (\omega_- - \omega_+^*) = i\gamma - (4\omega_0^2-\gamma^2)^{1/2}$$

$$[\quad] = \frac{1}{i\gamma} \frac{1}{(4\omega_0^2-\gamma^2)^{1/2}} \left[\frac{1}{A+iB} - \frac{1}{-A+iB} \right]$$

$$= \frac{1}{i\gamma} \frac{1}{\cancel{(4\omega_0^2-\gamma^2)^{1/2}}} \frac{\cancel{-2i\gamma} + \cancel{A}\times 2}{(4\omega_0^2-\gamma^2)+\gamma^2}$$

$$= \frac{(2\pi)|F|^2}{m^2\gamma} \frac{1}{(4\omega_0^2-\gamma^2)+\gamma^2}$$

$\gamma \ll \omega_0$

$$\omega_0^2 \int |x|^2 \frac{d\omega}{(2\pi)} = \frac{kT}{m}$$

$$\Rightarrow \boxed{|F|^2 = 2\, m\, kT\, \gamma}$$

Fluctuation dissipation theorem

$$|X(\omega)|^2 = \frac{2 m k T \gamma}{m^2\left[(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2\right]}$$

$$= \frac{2\gamma kT}{m\left[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right]}$$

$\omega \gg \omega_0$

$\gamma \ll \omega_0$

$$|X(\omega)|^2 \propto \frac{\gamma kT}{m\omega_0^4}$$

$$i\gamma\omega = k i \phi \quad \Rightarrow \phi = \frac{\gamma\omega}{m\omega_0^2}$$

$$\frac{\phi\,\omega_0^2}{\omega} = \gamma/m$$

$$\frac{4kT\gamma}{m^2} \quad \frac{1}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}$$

$$|X(\omega)|^2 = \frac{2\gamma kT}{m\left[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right]}$$

$\omega^2 \gg \omega_0^2$

$\gamma \ll \omega_0^2 \ll \omega^2$

$$|X(\omega)|^2 = \frac{2\gamma kT}{m\,\omega^4} = \left(\frac{kT}{m\omega^2}\right)\left(\frac{\gamma}{\omega_0}\right)\left(\frac{\omega_0}{\omega}\right)\frac{1}{\omega}$$

$$(\text{strain})^2 \propto \left(\frac{kT}{m\omega^2 L^2}\right)\underbrace{\left(\frac{\gamma}{\omega_0}\right)}_{1/Q}\underbrace{\left(\frac{\omega_0}{\omega}\right)}_{10}\underbrace{\frac{1}{\cancel{2}\omega}}_{10}$$

10^3 $\quad Q^{-1} \sim 10^{-6}$

$m \sim kg$ $\qquad T \sim 300K$

$\omega \sim 10\,s\,Hz$ $\qquad k \sim 1.4\;10^{-23}\;m^2\,kg\,s^{-2}\,K^{-1}$

$L \sim km$

$$\frac{kT}{m\omega^2L^2} \sim \frac{4\;10^{-21}}{10^2\;10^6} \sim 4\cancel{6}\;10^{-29} \times 10^{-9} \sim 10^{-38}$$

$P = 2\pi\sqrt{L/g}$ $\qquad \omega = 2\pi f = 2\pi\frac{1}{P}$

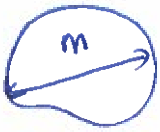
$\omega_0^2 = \frac{m}{10\,m/s^2}$

$\omega_0^2 = g/L \sim 10/s^2 \quad \rightsquigarrow \omega_0 \sim 3\,Hz$

Period ~ 1sec

Speed of sound in a solid — 6000 m/s

6 km/s



$\omega = c k$

$$\boxed{f = c_s/\lambda \sim \text{kHz}}$$

Journal of Physics E: Scientific Instruments

Limits to the measurement of displacement in an interferometric gravitational radiation detector

To cite this article: W A Edelstein *et al* 1978 *J. Phys. E: Sci. Instrum.* 11 710

fluctuation in displacement given by

$$(\delta x_A)^2 = \frac{F_A{}^2}{m^2\omega^4} = \frac{4I_0 h\nu\Delta f}{m^2\omega^4 c^2}.$$

There is a similar expression for m_B.

If we assume statistical independence of the fluctuations of the momenta and hence the displacement of the two mirrors (to be discussed below), we then find that the mean squared differential displacement is given by

$$(\delta x_2)^2 = \frac{8I_0 h\nu\Delta f}{m^2\omega^4 c^2}.$$

Note that this recoil effect becomes larger as I_0 is increased, which is as expected.

Any detectable displacement δx_d caused by external forces must be greater than that obtained by adding in quadrature δx_2 and the minimum value of δx_1 satisfying equation (1); that is

$$(\delta x_d)^2 \geqslant \frac{hc\lambda\Delta f}{8\pi^2 I_0} + \frac{8I_0 h\nu\Delta f}{m^2\omega^4 c^2}. \tag{2}$$

Adjusting I_0 to minimise the right-hand side of (2) yields

$$(\delta x_d)^2 \geqslant \frac{4\hbar\Delta f}{m\omega^2}. \tag{3}$$

3 Uncertainty principle limit

We now calculate the minimum observable differential displacement of m_A and m_B consistent with the uncertainty principle. Our treatment is similar to that of Braginskii and Vorontsov (1975).

Suppose our apparatus is capable of measuring the difference between the positions of two infinitely heavy masses in time t with an uncertainty of $(\Delta x)^2$ in the bandwidth $\Delta f = 1/2t$. Such a measurement would, however, produce an unpredictable change in the momenta of the masses, and for the finite masses in a real experiment, an unpredictable and possibly significant change in velocity; we now consider this effect.

Let Δp be the uncertainty in the momentum difference between masses m_A and m_B. The uncertainty relation $\Delta p\Delta x \geqslant \hbar$ (for two independent masses) implies a minimum momentum uncertainty given by

$$(\Delta p)^2 = \frac{\hbar^2}{(\Delta x)^2}.$$

In a bandwidth Δf, corresponding to an observing time $t = 1/2\Delta f$, this is equivalent to a differential force ΔF acting on the masses given by

$$(\Delta F)^2 = \frac{\hbar^2}{(\Delta x)^2 t^2} = \frac{4\hbar^2(\Delta f)^2}{(\Delta x)^2}.$$

Thus over a bandwidth Δf at an angular frequency ω well above the resonant frequency of the suspensions of the masses, this equivalent force would cause a differential displacement and hence a mean squared uncertainty in relative position

$$(\Delta x')^2 = \frac{4\hbar^2(\Delta f)^2}{m^2\omega^4(\Delta x)^2}.$$

The square of the total uncertainty δx_T in the differential displacement between m_A and m_B is given by the sum of $(\Delta x)^2$ and $(\Delta x')^2$, that is

$$(\delta x_T)^2 = (\Delta x)^2 + \frac{4\hbar^2(\Delta f)^2}{m^2\omega^4(\Delta x)^2},$$

and minimising with respect to $(\Delta x)^2$ we obtain

$$(\delta x_T)^2 = \frac{4\hbar}{m\omega^2}\Delta f.$$

which is the same as the minimum value of $(\delta x_d)^2$ satisfying expression (3).

4 Discussion of mass recoil

We now discuss our reasons for assuming statistical independence for the momentum fluctuations of the two masses.

Suppose a single photon has passed through the interferometer. The recoil effects can be seen as follows. There is a probability amplitude that the photon went into arm A, was reflected by the mirror attached to m_A, and caused m_A to recoil; there is a corresponding amplitude for arm B. The mirror test masses are then left in a superposition of states: in one state, m_A has received the entire impulse $2h\nu/c$ while m_B has received no impulse, and in the other state, the reverse has occurred.

In a multi-photon beam, the action of each photon can be considered approximately independent (Doppler effects can be shown negligible, and have been omitted). Then the effect of many photons is to leave the masses m_A and m_B in a quantum mechanical state with a probability distribution of momenta which is just as if each photon had randomly gone one way or the other.

When this effectively random division is combined with the fact that the incident photon distribution is Poisson, it can be shown that there is no correlation between the momentum fluctuations of m_A and m_B. The analysis is essentially the same as that which explains the lack of correlation between the photon number fluctuations in the two beams obtained from a beam splitter illuminated by a laser (see, for example, Loudon 1973, 1976).

As a final point, we remark that if the system was illuminated with chaotic light, there might be a correlated component in the momentum fluctuations of m_A and m_B; but if the system was carefully balanced, the differential recoil effects would be little worse than those obtained with coherent light.

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J R Pugh
W Martin

6 February 1978

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Binary system

$$M = m_1 + m_2 \qquad \eta = \frac{m_1 m_2}{(m_1 + m_2)^2} \qquad q = \frac{m_1}{m_2}$$

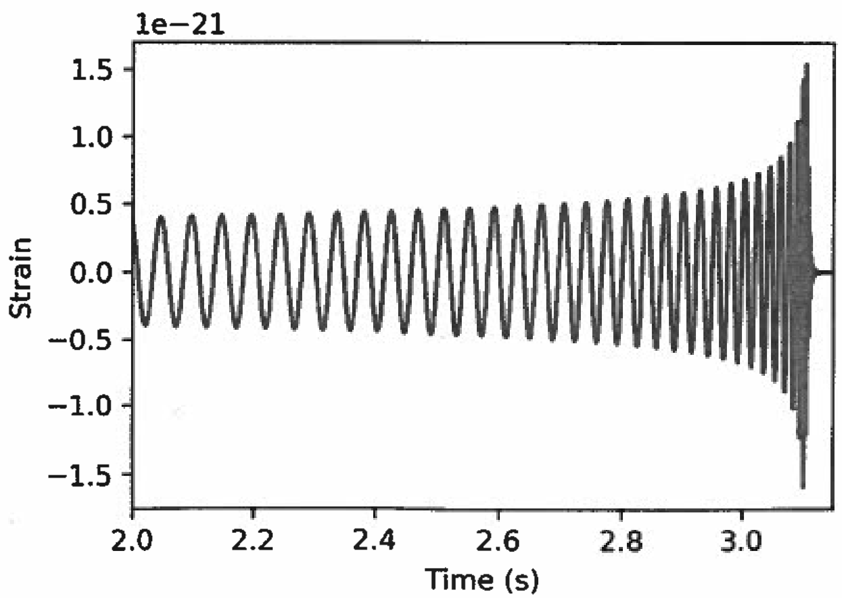
$$M_c = M\eta^{3/5} \qquad \omega_{\mathrm{GW}}(t) \sim M_c^{-5/8}\left(t_0 - t\right)^{-3/8}$$

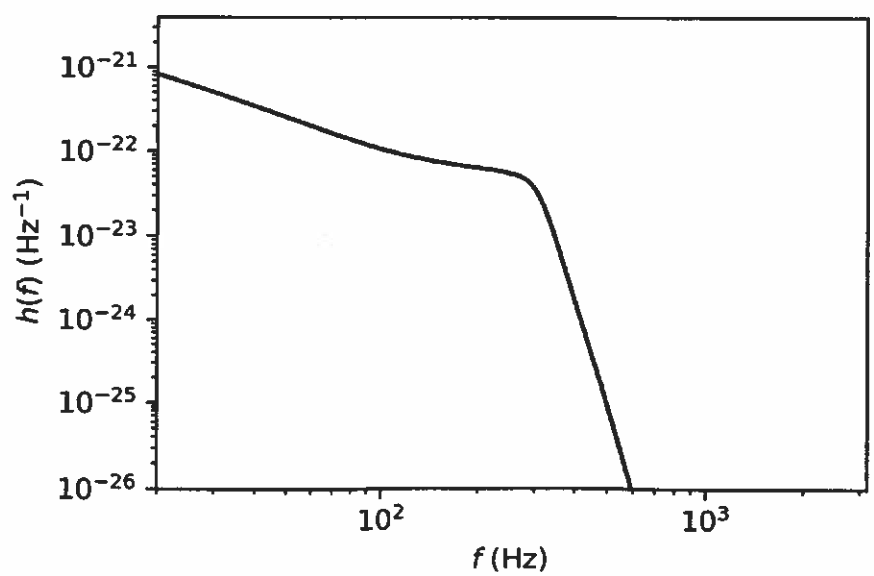
"Chirping" behavior

$$h(\omega_0) \propto M_c^{5/6} D^{-1} \omega_0^{-7/6} e^{i\phi(\omega_0)}$$

$$\phi(\omega_0) \propto (M_c\omega_0)^{-5/3}$$

$$\omega_{\max} \sim 100\,\mathrm{Hz}\left(\frac{50\,M_\odot}{M}\right)$$





Binary system

$M_T = (m_1 + m_2)$

$\eta = \frac{m_1 m_2}{M_T^2}$

$P_{orbit}^2 = (2\pi)^3 \frac{a^3}{M_T}$

$\Omega = \left(\frac{G M_T}{a^3}\right)^{1/2} = (M_T/a^3)^{1/2}$

$E = -\frac{1}{2} \frac{M_T^2 \eta}{a}$

$P = \frac{32}{5} \frac{M_T^5 \eta^2}{a^5}$

$$\frac{1}{a}\frac{da}{dt} = -\frac{1}{E}\frac{dE}{dt} = -\frac{1}{E}P = \frac{64}{5}\frac{M_T^3 \eta}{a^4} \Rightarrow a^4 \propto (M_T^3 \eta)(t_c - t)$$

$$a \propto (M_T^3 \eta)^{1/4} (t_c - t)^{1/4}$$

$$f = \frac{1}{P} \propto a^{-3/2} M_T^{1/2}$$

$$f \propto (M_T^3 \eta)^{-3/8} (t_c - t)^{-3/8} M_T^{1/2} \propto (M_T^{5/3} \eta)^{-3/8} (t_c - t)^{-3/8}$$

$$\propto \mathcal{M}_c^{-5/8} (t_c - t)^{-3/8}$$

$$\boxed{\mathcal{M}_c \equiv M_T \eta^{3/5}}$$

$$\boxed{f \propto \mathcal{M}_c^{-5/8} (t_c - t)^{-3/8}}$$

$$\frac{1}{f}\frac{df}{dt} \cancel{\propto} + \frac{3}{8}\frac{1}{(t - t_c)} \propto f^{8/3} \mathcal{M}_c^{5/3}$$

$$\boxed{df/dt \propto f^{11/3} \mathcal{M}_c^{5/3}}$$

$$\frac{d\phi}{dt} = f \qquad \frac{d\phi}{df} = \frac{d\phi}{dt}\left(\frac{df}{dt}\right)^{-1} = f/\frac{df}{dt} \propto$$

$$\Rightarrow \boxed{\phi \propto f^2/\frac{df}{dt} \propto \mathcal{M}_c^{-5/3} f^{-5/3} \propto (\mathcal{M}_c f)^{-5/3}}$$

$$h(t) \propto \frac{G M_T^2 \eta}{D r} \cos\phi(t)$$

$$\int e^{ift} h(f) \propto \frac{G M_T \eta}{D} \int e^{-ift} e^{i\phi(t)} dt$$

$$\psi(t) = -ft + \phi(t)$$

$$\phi(t) = \phi_{t_0} + \left.\frac{d\phi}{dt}\right|_{t_0} (t-t_0)^2 + \frac{1}{2}\frac{d^2\phi}{dt^2}()^2$$

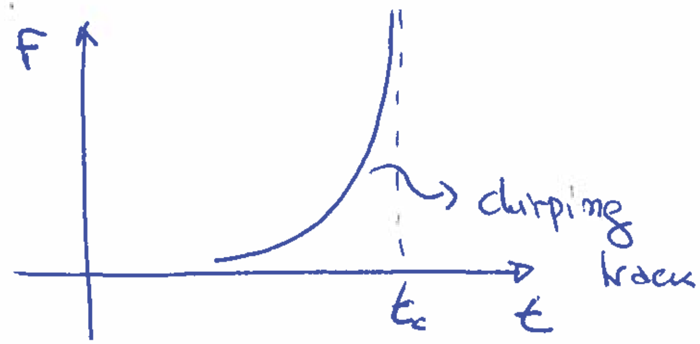
$$\frac{\partial\psi}{\partial t} = 0 \Rightarrow \frac{d\phi}{dt} = f \qquad A = M_T \eta f^{11/6}$$

$$\psi = \phi_0 + \frac{1}{2}\frac{df}{dt}(t-t_0)^2$$

$\rightarrow$ Gaussian Integral $\propto \left|\frac{df}{dt}\right|^{-1/2}$

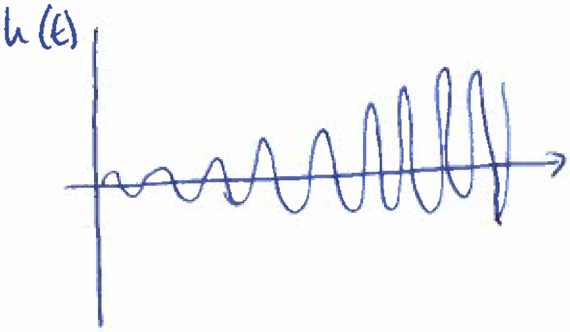
$$\Rightarrow \cancel{h(f) \propto \frac{M_T \eta}{D}\left(\frac{df}{dt}\right)^{-1/2} e^{i\phi(f) + f t_0}} \qquad \boxed{h(f) \propto \frac{f^{-7/6} \mathcal{M}_c^{5/6}}{D} e^{i\Psi}}$$

$$\Psi = 2\pi f t_c - \phi_c + \frac{3}{4}(8\pi \mathcal{M} f)^{-5/3}$$



$$F \propto (t_c - t)^{-3/8}$$

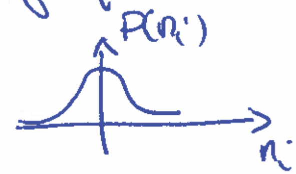
$$A(f) \propto f^{-7/6}$$

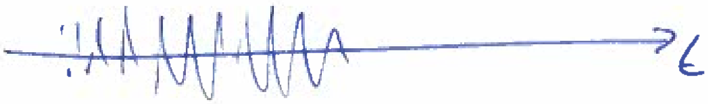


Searching for signal

In the absence of signal

$d_i = n_i$





What is the probability of getting a given "curve" n_i JUST FROM NOISE?

- Ideal experiment

$$\langle n_i n_j \rangle = \langle n(t_1) n(t_2) \rangle = C(t_1 - t_2)$$

stationary noise

$$\langle n_i \rangle = 0$$

$$n(f) = \int dt \, e^{-ift} n(t)$$

$$\langle n^*(f_1) n(f_2) \rangle = \int dt_1 \, e^{+if_1 t_1} \int dt_2 \, e^{-if_2 t_2} \underbrace{\langle n(t_1) n(t_2) \rangle}_{C(t_1 - t_2)}$$

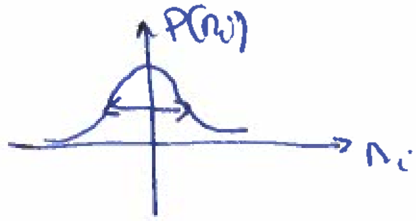
$$t_1 = t_2 + \Delta t$$

$$= \underbrace{\int dt_2 \, e^{i(f_1 - f_2) t_2}}_{\propto \delta^D(f_1 - f_2)} \underbrace{\int dt_1 \, e^{i f_1 \Delta t} C(\overset{\Delta t}{t_1 - t_2})}_{P(f_1)}$$

$$\boxed{\langle n(f_1) n(f_2) \rangle \propto \delta^D(f_1 - f_2) P(f_1)}$$

stationary noise $\Rightarrow$ Diagonal in Fourier space

Stat Gaussian Noise

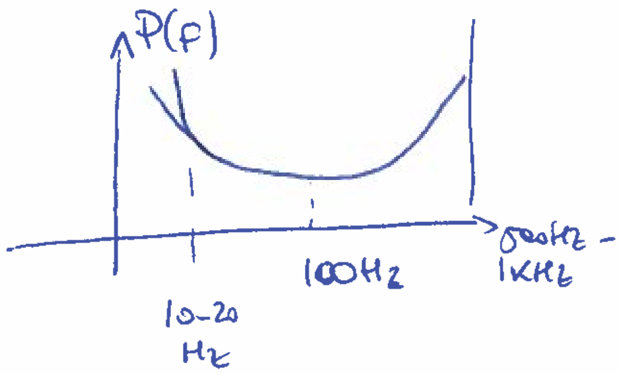


$$P(\tilde{n}|N) \propto \frac{\exp\left(-\frac{1}{2}\, n N^{-1} n\right)}{(\det N)^{1/2}}$$

1) Go to Fourier space $\quad \langle \tilde{n}_i \tilde{n}_j \rangle = \frac{\delta_{ij}}{\Delta f} P(f)$

$\tilde{n}_i$ = ~~A~~ Noise for each frequency $\quad \Delta f \propto \frac{1}{T}$

$$P(\tilde{n}) = \prod_f \exp\left(-\frac{1}{2} \frac{|n_f|^2}{P(f)}\right) \qquad \sigma^2(f) \propto P(f)/\Delta f$$



$\frac{n_f}{\sigma_f}$ = weighted data → has noise independent of frequency

If weighted data first:

$N\big|_{\text{Fourier space}} = \delta_{ij} \leftrightarrow \delta_{ij}$ in real space.

For convenience all mostly do that not to carry around N

$d = \underbrace{A\tilde{h}}_{h} + n$ Hipothesis.

Is h in the data

If $h=0 \rightarrow d$ is n

otherwise $d = h + n \Rightarrow n - (d-h)$

Define

$$\langle f_1 | f_2 \rangle = \sum_f \frac{f_1^* f_2}{\sigma^2}$$

$\equiv$ DOT product between f_1 f_2

$$\frac{\langle f_1 | f_2 \rangle}{\langle f_1 | f_1 \rangle^{1/2} \langle f_2 | f_2 \rangle^{1/2}} = \gamma$$

$$P(n|H_1) \propto \prod_{freq} \exp -\frac{1}{2}\frac{\|d - Ah\|^2}{\sigma^2} \propto \exp\left(\|d-h\|^2\right) \propto \left(\langle d|d\rangle + \langle h|h\rangle - 2\,\mathrm{Re}\,\langle d|h\rangle\right)$$

$$P(n|H_0) \propto \prod_{freq} \exp\left(-\frac{1}{2}\|d\|^2/\sigma^2\right)$$

$$\Rightarrow \ln \frac{P(d|H_1)}{P(d|H_0)} \propto -\frac{1}{2}\left(\frac{\langle d|d\rangle + \langle h|h\rangle - 2\,\mathrm{Re}\,\langle d|h\rangle}{\cancel{\langle d|d\rangle}} - \langle d|d\rangle\right)$$

$$\ln P_1/P_0 \propto -\frac{1}{2}\left(\langle h|h\rangle - 2\,\mathrm{Re}\,\langle d|h\rangle\right)$$

$$\langle d|h\rangle = \sum_f \frac{h^* d}{\sigma^2}$$

Amp $\quad Z = \sum_f \frac{\tilde{h}^* d}{\sigma^2}$ $\quad$ Normalize $\sum \frac{|h|^2}{\sigma^2} = 1$

$$\boxed{\ln P_1/P_0 = -\frac{1}{2}\left(|A|^2 - 2\,\mathrm{Re}\,AZ\right)}$$

$$= -\frac{1}{2}\ \|A - Z\|^2 + \frac{\|Z\|^2}{2}$$

Maximize over A

$$\Rightarrow \boxed{\ln P_1/P_0\Big|_{A_{max}} = \|Z\|^2/2}$$

If d is just noise

$$Z = \sum_f \frac{\tilde{h}^* d}{\sigma^2}$$ is a random variable distributed like a Gaussian

Real & Imaginary parts.

$$\langle Z \rangle = 0$$

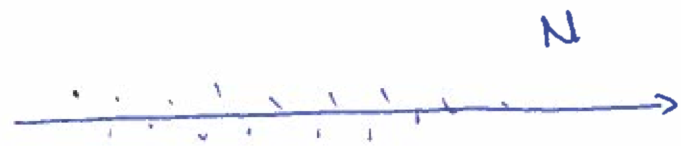
$$\langle Z^* Z \rangle = \sum_{f_1} \sum_{f_2} \frac{\tilde{h}^*(f_1)\, \tilde{h}(f_2)}{\sigma^2(f_1)\, \sigma^2(f_2)} \underbrace{\langle \tilde{n}(f_1)\, n(f_2) \rangle}_{\delta_{f_1 f_2} \sigma^2}$$

$$= \sum_f \frac{|\tilde{h}|^2}{\sigma^2} = 1$$

$$\Rightarrow P(Z | H_0) \propto \exp\left(-\frac{\|Z\|^2}{2}\right)$$

$$Z = \sum_f \frac{\tilde{h}^*(f)\, d(f)}{\sigma^2}$$

Search statistic



$\mathbb{R}^N$

Can able to describe function

however I want

- Time domain
- Fourier domain

- First vector : waveform (normalize vec form) (actually 2) for sine & cosine.

orthogonal space $N-2$

$\vec{v}_0, \vec{v}_1$ $\quad$ $\left.\begin{matrix}\vec{v}_0 \cdot \vec{d} \\ \vec{v}_1 \cdot \vec{d}\end{matrix}\right\}$ each is a Gaussian random variable

$$\text{Model} = \alpha_0 \vec{v}_0 + \beta_0 \vec{v}_1$$

$$\ln L = \frac{1}{2}\left[(\alpha_0 - \hat{v}_0 \cdot \vec{d})^2 + (\alpha_1 - \hat{v}_1 \cdot \vec{d})^2\right] + \sum_{k>1} (\hat{v}_k \cdot \vec{d})^2$$

Maximize $\ln L$ by picking $\alpha_0 = \hat{v} \cdot \vec{d}$

$$\Delta \ln L = \frac{1}{2}\left[(\hat{v}_0 \cdot \vec{d})^2 + (\vec{v}_1 \cdot \vec{d})^2\right]$$

How do I search

$\tilde{h}(f)$ depends on the time of merger & chirp mass & other parameters $= \vec{\theta}$

$$\frac{P(d|\vec{\theta}_1)}{P(d|\vec{\theta}_2)} = \frac{P(d|\vec{\theta}_1)}{P(d|H_0)} \quad \frac{P(d|H_0)}{P(d|\vec{\theta}_2)}$$

$$\ln P_1/P_2 \propto \left(\|Z_1\|^2 - \|Z_2\|^2 \right)$$

This is how you do parameter estimation as well.

$$\boxed{\text{Vary parameters and try to get largest } \|Z\|^2}$$

- How likely it is that you get that $\|Z^*\|^2$ from noise?

- Same procedure to fit parameters.
  Find highest $|Z|^2$ and then see how far you can move to

$h(f) = A\, e^{i\Psi} = A(f)\, e^{i\Psi}$ $\qquad A(f) \propto f^{-7/6}$

$\Psi = 2\pi f t_c - \phi_c + \frac{3}{4}(8\pi\mu)^{-5/3} f^{-5/3}$ $\qquad A(f)\, e^{i\phi_c}$ complex #

$\Psi = -\phi_* + \cancel{2\pi f t} \; 2\pi(f - f_*) t_c + \frac{3}{4}(8\pi\mu)^{-5/3}(f^{-5/3} - \alpha f - \beta)$

Making each component orthogonal

$\Psi = \sum_\alpha c_\alpha \Psi_\alpha$ $\qquad \Psi_0 = $ constant

$\Psi_1 = mf + b$ linear funct

$\Psi_2 = A f^{-5/3} + Bf + C$

$$\int_f df \, \Psi_\alpha \Psi_\beta \frac{|h(f)|^2}{\sigma^2(f)} = \delta_{\alpha\beta}$$

$\frac{\partial h}{\partial c_\alpha}$ are all orthogonal.

$\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4$

Amplitude & phase, time, chirp mass, spin

$|z|^2 \to \chi^2$ 5 degree of freedom.

If you go too far away in params ⇒ already orthogonal

Fisher matrix

$$d = A_0 h_0 + n$$

$$Z = \sum_f A_0 \tilde{h} h_0 + \sum \tilde{h} n$$

$$\|Z\|^2 = \sum_{f_1} \sum_{f_2} |A_0|^2 \tilde{h}^*(f_1) h_0(f_1) \ \tilde{h}(f_2) h_0 f$$

$$\sum_f \frac{\tilde{h}^* h_0}{\sigma^2} = \mu$$

$$Z = A\mu$$
$$|Z|^2 = |A|^2 \mu^2$$

$$\tilde{h} = h_0 + \frac{\partial h_0}{\partial \theta_i} \Delta\theta_i + \frac{1}{2} \frac{\partial^2 h_0}{\partial \theta_i \partial \theta_j} \Delta\theta_i \Delta\theta_j$$

$$\Delta |Z|^2 =$$
$$|Z|^2_{opt} (\mu^2 - 1)$$

$$\mu = 1 + \Delta\theta_i \cancel{\sum_f \frac{\partial h^*}{\partial \theta_i} h_0} + \frac{1}{2} \left(\sum_f \frac{\partial^2 h}{\partial \theta_i \partial \theta_j} h \right) \Delta\theta_i \Delta\theta_j$$

significance

$$\propto e^{-|Z|^2}$$
$$\propto e^{-|Z|^2_{opt} (\mu^2 - 1)}$$

$$\|Z\|^2 - 1 = \mu^2 - 1$$

$$\frac{1}{2} \Delta\theta_i \frac{\partial}{\partial \theta} \sum_f \hat{h}_0 h \simeq 0$$

$$\sum_f \frac{\partial^2 h}{\partial \theta \partial \theta} h$$

$$\frac{\partial}{\partial \theta} \left[\frac{\partial h}{\partial \theta} h - \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta} \right]$$

$$\frac{\partial}{\partial \theta} \left[\frac{1}{2} \frac{\partial}{\partial \theta} h^2 - \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta} \right]$$

$$\left. \begin{aligned} \mu &= 1 - \frac{1}{2} \left(\sum_f \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta} \right) \Delta\theta \Delta\theta \\ \mu^2 &= 1 - \left(\sum_f \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta} \right) \Delta\theta \Delta\theta \end{aligned} \right] \quad Z^2 = \left(\sum_f \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta} \right) \Delta\theta^2$$

$$\langle \ln L \rangle = -\frac{1}{2} \Delta\theta_i \Delta\theta_j F_{ij}$$

$$F_{ij} = \sum_f \frac{\partial h^*}{\partial \theta_j} \frac{\partial h}{\partial \theta_i} \frac{1}{\sigma^2}$$

$$h = A \tilde{h}$$

$$= A \exp\left[i\phi_0 + i f t_c + \frac{i}{\cancel{(M_c f)}} (f M_c 8\pi)^{-5/3} \, 3/4\right]$$

$$\frac{\partial h}{\partial A} = \tilde{h}$$

$$\frac{\partial h}{\partial t_c} = i f \tilde{h}$$

$$\frac{\partial h}{\partial M_c} = i \frac{3}{4} (8\pi f M_c)^{-5/3} \frac{(-5/3)}{M_c}$$

$$\frac{\partial h}{\partial \ln M_c} = -i \frac{5}{4} (8\pi f M_c)^{-5/3}$$

$$F_{AA} = 1$$

$$F_{t_c t_c} = \sum_f f^2 \frac{|\tilde{h}|^2}{\sigma^2}$$

$$F_{\ln M \ln M} \propto \sum_f \frac{|\tilde{h}|^2}{\sigma^2} (Mf)^{-10/3}$$

$$\Delta \ln M \propto \frac{M^{5/3}}{(SNR)}$$

When can you claim a detection?

$$P(z^2) \propto e^{-|z|^2/2} \times N_{trials}$$

$$N_{trials} = N_{templates} \times N_{times}$$

$$N_{times} = \frac{T}{10 \text{ ms}} \qquad T \sim 3\,10^7 \text{ sec}$$

$$N_{times} \sim 10^{10}$$

$$N_{templates} \propto 10^3$$

$$N_{trials} \sim 10^{13}$$

$$|z|^2 \sim 2 \ln N_{trials}$$

$$\rho \sim (|z|^2)^{1/2} \propto \sqrt{2 \ln N_{trials}} \sim 7$$

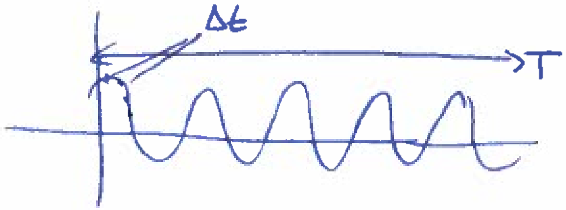
Non-ideal issues: - distribution is not a Gaussian.

- Multiple detector
- Time slides

- Not stationary
- Solution is parameter dependent

Toy model

$$s_i = A\cos(\omega t_i) + n_i$$



n_i = Gaussian white noise

$$\langle n_i n_j \rangle = \sigma_n^2 \, \delta_{ij}$$

$$N = T/\Delta t \qquad \sigma_n^2 = \frac{[illegible]}{\Delta t}$$

want to combine all the data points

1) A can be positive or negative, signal averages to 0. → need to square.

$$X = \sum_{i=1}^{N} s_i^2$$

$$= A^2 \sum_i \cos^2 \omega t_i + 2A \sum_i \cos \omega t_i \, n_i + \sum_i n_i^2$$

$$\langle X \rangle = N \frac{A^2}{2} + N \sigma_n^2$$

$$\tilde{X} = \sum_i \left(s_i^2 - \sigma_n^2 \right) \qquad \langle \tilde{X} \rangle = N \frac{A^2}{2}$$

$$= A^2 \sum_i \cos^2 \omega t_i + \underbrace{2A \sum_i \cos \omega t_i \, n_i + \sum_i \left(n_i^2 - \sigma_n^2 \right)}_{\text{what is the variance?}}$$

ϵ

$$\epsilon^2 = 4A^2 \sum_{ij} \cos\omega t_i \cos\omega t_j \, n_i n_j + 4A \sum_i\sum_j \cos\omega t_i \, n_i (n_j^2 - \sigma_n^2)$$

$$+ \sum_i\sum_j (n_i^2 - \sigma_n^2)(n_j^2 - \sigma_n^2)$$

$$\langle \epsilon^2 \rangle = 4A^2 \sigma_n^2 N + 2\sigma_n^4 N$$

$$\tilde{X}' = \frac{1}{N} \sum_i (s_i^2 - \sigma_n^2) \qquad \langle \tilde{X} \rangle = A^2/2$$

$$\tilde{X} = \langle \tilde{X} \rangle + \varepsilon \qquad \sigma_\varepsilon^2 = \frac{2\sigma_n^2}{\sqrt{N}} + \frac{2A\sigma_n}{\sqrt{N}}$$

At detection limit

$$\frac{A^2}{2} \sim \frac{2\sigma_n^2}{\sqrt{N}} \Rightarrow A \sim \frac{2\sigma_n}{N^{1/4}}$$

~~For N>>1~~

$$\tilde{X}\hat{A} > \frac{\langle \tilde{X} \rangle^2}{\langle \epsilon^2 \rangle} = \cancel{\frac{N^2 A^4}{4\sigma_n^4/N}}$$

$$= \left(\frac{A^2}{2}\right)^2 \frac{N}{2\sigma_n^4} = \frac{N}{8} \left(\frac{A^2}{\sigma_n^2}\right)^2$$

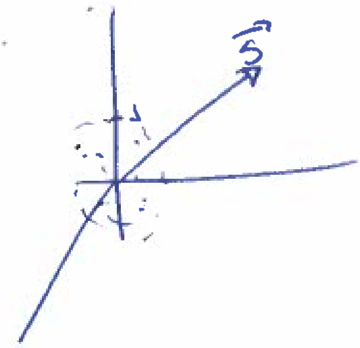
$$y = \frac{1}{N}\sum_i \cos\omega t_i \, s_i = \frac{A}{2} + \underbrace{\frac{1}{N}\sum_i \cos\omega t_i \, n_i}_{\epsilon}$$

$$\langle y \rangle = A/2$$

$$\langle \epsilon \rangle = 0 \qquad \langle \epsilon^2 \rangle = \frac{1}{N}\frac{\sigma_n^2}{2}$$

$$(S/N)^2 = \boxed{\frac{\langle y^2 \rangle}{\langle \epsilon^2 \rangle} = \frac{N}{2} A^2/\sigma_n^2}$$

$$(S/N)^2_{\tilde{x}} \Big|_{(S/N)^2|_y = 1} = \frac{N}{8}\left(\frac{A^2}{\sigma_n^2}\right)^2 \propto N \frac{1}{N^2} \propto 1/N$$



$$L^2 = \frac{A^2}{2} \times N$$

If rotate so $\vec{s}$ is along axis = 1

orthogonal transformation

⇒ each direction has the same variance = 1

$$\Rightarrow \boxed{\frac{L^2}{\sigma^2} = \left(\frac{S}{N}\right)^2 = \frac{A^2}{2} N}$$

~~What~~ How do I get the length

$\hat{L} = \vec{s}\cdot\vec{d}$ $\quad \cancel{\langle s_i s_j \rangle}$ $|\vec{s}|^2 = 1$

$s_i = \sqrt{\frac{2}{N}} \cos\omega t_i$ $\quad \vec{s}\cdot\vec{d} = A + \sqrt{\frac{2}{N}}\sum_i \cos\omega t_i \, n_i = \hat{L}$

$$\langle \hat{L} \rangle = A$$

$$\langle \epsilon^2 \rangle = \frac{2}{N}\sum_i \cos^2\omega t_i = 1$$

each individual line

S/N $\quad s^2 = A^2/2$

Computational cost

4096 sec ~ abt over 1 hour

200 days ~ 200 x 24 hrs

~ 5 10^3 hrs.

Few thousand files

Cost of convolution.

N ln N operations   N is # of pt samples

sampling of the data at 1024 Hz

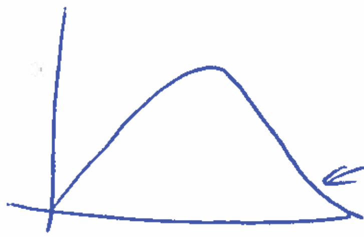
$N \sim 1024 \times 4096 \sim 4 \times 10^6$ samples

$5 \times N \log_2 N \sim 4\ 10^8$ operations   seconds

10^3 templates $\times 10^3$ files $\sim 10^6$ seconds

Z^2 is χ^2 with 5 degrees of freedom

$P(Z^2)$



$\propto |Z|^4 \, e^{-|Z|^2/2}$

Do Not pays this if working off other triggers

trials Factor $\sim 10^2$ Templates $\times \dfrac{(120 \text{ days} \times 24 \text{ hrs} \times 3600 \text{ sec})}{1 \text{ ms}}$

$\sim 10^{12}$ ← big price.

$P(Z^2)$ Trials ~ 1 at this $|Z|^2$ you expect 1 FALSE ALARM per O2 run.

$\Rightarrow \boxed{Z^2\big|_{FAR\sim 1} = 66}$

$\rho_{Ideal} \sim 8$

Maybe more typical is $\rho \sim 9$

$\rho = \left(\int h^2/\sigma^2\right)^{1/2} \propto 1/D \Rightarrow D \propto 1/\rho_{cut} \quad V \propto \dfrac{1}{\rho_{cut}}$

$\frac{\text{Trials}}{\text{epoch}} \propto T$

$Z_{cut} \propto \sqrt{\ln \frac{\text{Trials}}{\text{epoch}}}$ $\Rightarrow$ Do not pay as much for $T \uparrow$ as you will

$N_{expected} \sim R\,V\,T$

150914 SNR ~ 15

$M_T \sim 70\, M_\odot$

$\mathcal{M}_c \sim 30\, M_\odot$

$D \sim 400$ Mpc

$z \sim 0.1$

Number of templates:

BNS $\sim 10^5$

	Mchirp	
BBH 1	5 10	5000
BBH 2	10 20	2000
BBH 3	20 40	200
BBH 4	>40	50