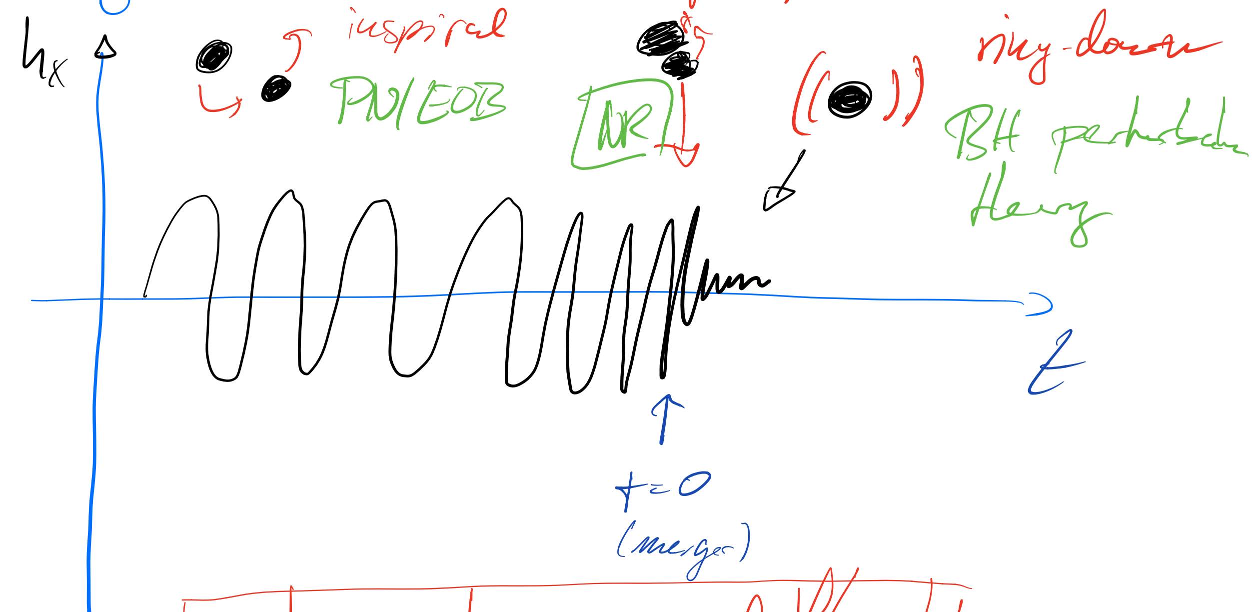
Based on Baumgarte & Shapiro, "Numerical relativity", 2010

Why do we need numerical relativity?

Binary black hole coalescence



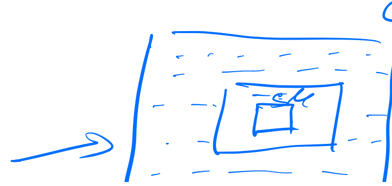
Neutron stars are different!

NR is needed for the dynamical part at merger and for cross-checking PN/EOB expressions.

Why not always use just NR?
Do we really need to do all of these complicated PN integrals?

Let's estimate

computational grid



1×100^3 per grid $\quad$ $\sim 1000M$

pure computational cost ($\sim 10^4$ FLOPS)

~ 1 TFlop/s $\rightarrow 10^9$ FLOPS$_s$

timestep $\sim 0.3 \times 10^{-3}$ $\qquad t_f \sim 10^3$

$\Rightarrow 3 \times 10^6$ timesteps

$$\Rightarrow \frac{7 \times 10^3 \times 10^4 \times 3 \times 10^6}{10^9} s = 2 \times 10^5 s$$

$1 hr \sim 3 \times 10^3 s$

≈ 100 hrs

$\Rightarrow$ How long does EOB/IPhenom take?

$\Rightarrow < 1s$

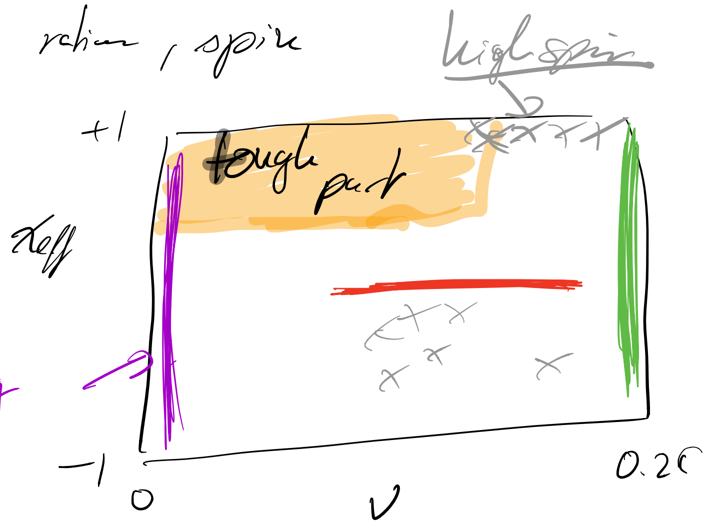
Can run millions of waveforms while waiting for one NR simulation

If so expensive, what does know?

Largest catalog SXS − 2019

→ ~ 2000 simulations BBH

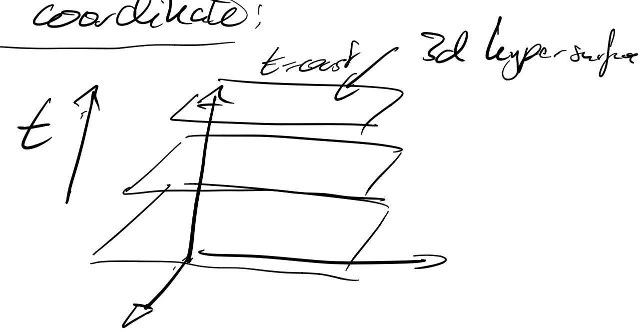
precession, mass ratio, spin



tough part: $\chi > 0.8$, > 50 cycles, $q > 4$

How do simulations work?

We need a time coordinate:



A hypersurface is uniquely specified by $t(x^\mu) = \text{const}$

$$\Leftrightarrow \quad \nabla_\mu t(x^\nu) = \Omega_\mu$$

can normalize

$$\boxed{n^\mu = -g^{\mu\alpha} \frac{\Omega_\alpha}{\sqrt{\Omega_\beta \Omega^\beta}}}$$

normal vector n^μ of an Eulerian observer

Induced metric on the hypersurface

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

$$\Rightarrow \quad n^\mu \gamma_{\mu\nu} = 0$$

We can use this to define derivatives on the surface

$$\Rightarrow \quad D_a f := \gamma_a{}^\mu \nabla_\mu f$$

$$D_a v_b = \gamma_a{}^\mu \gamma_b{}^\nu \nabla_\mu v_\nu$$

With this definition

$$^{(3)}\Gamma_{bc}{}^a = \frac{1}{2}\gamma^{ad}\left(\partial_c \gamma_{db} + \partial_b \gamma_{dc} - \partial_d \gamma_{bc}\right)$$

Are we good here? NO – we seen from previous DOF counting that there should be 4 gauge degrees of freedom

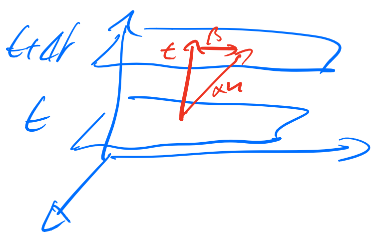
We just fixed 1.

What goes wrong?

$n^\mu \Omega_\mu = n^\mu \nabla_\mu t \neq 1$ So this is not the natural time direction!

Rather $t^\mu = \alpha n^\mu + \beta^\mu$ is

$$\Rightarrow \quad t^\mu \Omega_\mu = \alpha n^\mu \Omega_\mu + \beta^\mu \Omega_\mu = 1$$



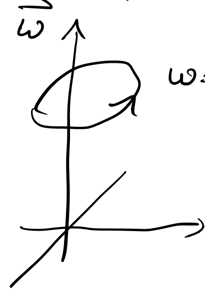
$$\boxed{\begin{aligned} ds^2 = &-dt^2(\alpha^2 - \beta^i\beta_i) \\ &+ \beta_i dx^i dt \\ &+ \gamma_{ij} dx^i dx^j \end{aligned}}$$

3+1 metric.

How do we understand the shift?

Imagine we were uniformly rotating:

$\vec{v} = \vec{\Omega} \times \vec{r}$



$\omega = \frac{d\varphi}{dt} = \text{const.}$

$\Downarrow$

translate this to a four vector:

$$\boxed{v^i = \frac{\gamma^i{}_\mu u^\mu}{-n_\mu u^\mu}}$$

$$\Rightarrow \quad \boxed{\frac{u^i}{u^0} = \alpha v^i - \beta^i}$$

Flat space $\alpha = 1$, $\beta^i = 0$ $\frac{u^i}{u^0} = v^i = \underline{\underline{\varepsilon^{ijk}\Omega_j x_k}}$

How about this: $\beta^i = -\varepsilon^{ijk}\Omega_j x^k$, $\tilde{v}^i = 0$

$$\Rightarrow \frac{\tilde{u}^i}{u^0} = \varepsilon^{ijk}\Omega_j x_k = \frac{u^i}{u^0}$$

A massless particle would move the same, but the interpretation is much different.

In the first instance, the particle moves, in the second the coordinates do!

$\boxed{\text{However}}$ $ds^2 = +dt^2(\beta^2 - 1) + \beta_i dx^i dt + \gamma_{ij} dx^i dx^j$

<u>still flat</u>

<u>Exhibit curvature</u>

Now we have a like:

Think about Newtonian gravity

$$\ddot{x}^i = -\frac{GM}{|x|^3} x^i$$

$$\Rightarrow \frac{dx^i}{dt} = v^i, \quad \frac{dv^i}{dt} = -\frac{GM}{} x^i$$

$$ \quad \quad / \quad dt \quad \quad |x|^3 $$

$\Rightarrow$ Need something like $\underline{\partial_t \gamma_{\mu\nu}}$!

Intuitively this should encode how γ_{ij} changes between hypersurfaces

$$\Rightarrow \boxed{-\frac{1}{2} \mathcal{L}_n \gamma_{ab} =: K_{ab}}$$

$$\boxed{\mathcal{L}_X T^a{}_b = X^c \partial_c T^a{}_b - T^c{}_b \partial_c X^a + T^a{}_c \partial_b X^c}$$

$$\Rightarrow \boxed{\mathcal{L}_t \gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta \gamma_{ab}}$$

$D_i \beta_j + D_j \beta_i$

How does physics enter the picture?

The foliation has to be consistent with the 4d manifold!

$$\mathcal{L}_n K_{ab} = n^d n^c \gamma^q{}_a \gamma^r{}_b \, {}^{(4)}R_{drcq} - \frac{1}{\alpha} D_a D_b \alpha - K^c{}_b K_{ac}$$

physics enters here

$\boxed{\text{Dynamical DOFs}}$

We also have

Hamiltonian constraints

$$\boxed{{}^{(3)}R + K^2 - K_{ab}K^{ab}} = \gamma^{pr}\gamma^q{}_b\gamma^s{}_a \, {}^{(4)}R_{pqrs}$$

$$\boxed{16\pi \, n_\mu n_\nu T^{\mu\nu}}$$

Compare with $\boxed{\nabla \cdot E = 4\pi q}$ in electrodynamics

$$D_b K^b{}_a - D_a K = 8\pi S_a \qquad \boxed{\nabla \cdot B = 0}$$

Constructing initial data for NR is hard! → These are all elliptical problems. We could spend two more lectures on how this is done.

We are good to go, aren't we?

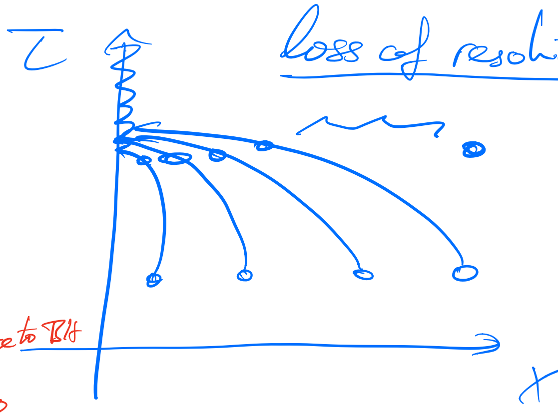
NOT YET!

Need to choose valid gauge.

Potential problems black hole

Choose $\alpha = 1$, $\beta^i = \vec{0}$

$\Rightarrow$ coordinates move at $u^\mu = t^\mu$



Intuitively: $d\tau = \alpha \, dt$, $\alpha \to 0$ close to BH

$\Delta t = \int \frac{1}{\alpha} d\tau \to \infty$

We don't want all of our numerical grid to end up in the black hole!

We can measure this "expansion" in terms of fluid divergence

$$\nabla_\mu u^\mu \hat{=} \nabla_\mu n^\mu = g^{\mu\nu} \nabla_\nu n_\mu = -K$$

def. K

so for $K>0$ (i.e. BH) will shrink to zero!

Solution $\boxed{K=0}$ (maximal slicing)

$\Rightarrow \partial_t K = 0$

$$\Rightarrow \boxed{D^2\alpha = \alpha\left(K_{ij}K^{ij} + 4\pi(\rho + S)\right)}$$

This is an elliptic problem. (Expensive!)

Harmonic gauge

$${}^{(4)}\Gamma^\lambda = g^{\mu\nu}\,{}^{(4)}\Gamma^\lambda_{\mu\nu} = -\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\lambda}\right)$$

Harmonic

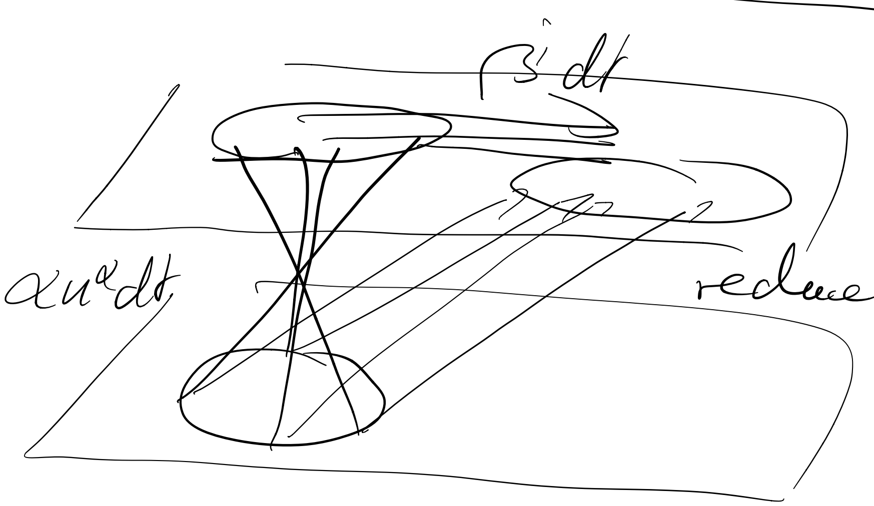
$$\boxed{{}^{(4)}\Gamma^\lambda = 0} \Leftrightarrow \Box x^\mu$$

$$\Rightarrow \boxed{\partial_t\alpha = \beta^j\partial_j\alpha - \alpha^2 K}$$

damps the lapse!

$$\Rightarrow \boxed{\partial_t \alpha = -\alpha^2 f(\alpha) K \qquad f(\alpha) = \frac{2}{\alpha}}$$

$$\Rightarrow \boxed{\mathcal{L}_n \alpha = -2K}$$



reduce "shear" of coordinates

$$u_{ij} = \gamma^{1/3} \partial_t (\gamma^{-1/3} \gamma_{ij})$$

$$A = \int u_{ij} u^{ij} \sqrt{\gamma}\, d^3x \quad \leftarrow \text{shift minimizes this}$$

$$\Rightarrow \boxed{D^i u_{ij} = 0}$$

$$\Rightarrow \partial_t \bar{\Gamma}^i = 0 \qquad \bar{\Gamma}^i = -\partial_j \bar{\gamma}^{ij}$$

$$\Rightarrow \partial_j (\bar{u}^{ij}) = 0$$

$$\boxed{\partial_t \beta^i = (\partial_t \bar{\Gamma}^i + \eta \bar{\Gamma}^i)}$$

$$\partial_t \beta^i = \frac{3}{4} B^i + \beta^j \partial_j \beta^i$$

$$\partial_t B^i = \beta^j \partial_j B^i + \partial_t \bar{\Gamma}^i - \eta B^i$$

Gamma driver

Minkowski

$$\partial_t A_i = -E_i - D_i \Phi$$

$$\partial_t E_i = -D^j D_j A_i + D_i D^j A_j - 4\pi j_i$$

Wave operator!!

not so cool.

Get a wave equation for A_i if
$D_i A = 0$ & $\phi = 0$ (Coulomb gauge)

What happens if this is violated?

$\Gamma = D_i A^i$

Define: $\partial_t \Gamma = \partial_t D^i A_i = D^i \partial_t A_i$

$= - D^i E_i - D_i D^i \Phi = - D_i D^i \Phi$
$- 4\pi q$

$$\partial_t E_i = - D_j D^j A_i + D_i \Gamma - 4\pi j_i$$

Mixed derivative is gone!

$$R_{ij} = \frac{1}{2} \gamma^{kl} \left(\partial_l \partial_i \gamma_{kj} + \partial_j \partial_k \gamma_{il} - \partial_j \partial_i \gamma_{kl} - \partial_l \partial_k \gamma_{ij} \right) + \gamma^{kl} \left(\Gamma^m_{il} \Gamma_{mkj} - \Gamma^m_{ij} \Gamma_{mkl} \right)$$

wave operator

BH $\boxed{R_{\mu\nu} = 0}$

Can fix this if ${}^{(4)}\Gamma^a = H^a(h, x)$

generalized harmonic gauge

One more ingredient:

Think about Schwarzschild

$$ds^2 = -\left(1 - \frac{2M}{R}\right)dt^2 + \frac{1}{\left(1-\frac{2M}{R}\right)}dr^2 + r^2 d\Omega^2$$

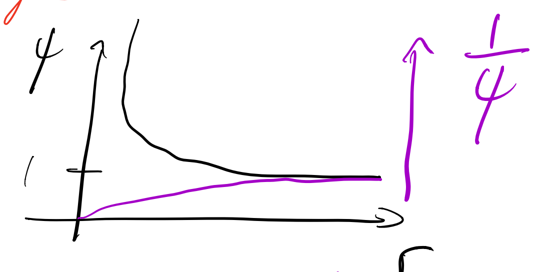
Can recast this as

$$ds^2 = -\left(\frac{1 - M/2r'}{1+M/2r'}\right)^2 dt^2 + \underbrace{\left(1+\frac{M}{2r'}\right)^4 \left(dr^2 + r^2 d\Omega^2\right)}$$

$$\boxed{\gamma_{ij}\,dx^i dx^j = \psi^4 \gamma_{ij}\,dx^i dx^j}$$

The three metric of a Schwarzschild BH is <u>confromally flat!</u>

So as $r' \to 0$



So e.g. ψ^{-2} remains <u>regular</u> at the origin

So for numerical evolutions split

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij} \,, \quad \det \bar{\gamma}_{ij} = 1$$

$$K_{ij} = \psi^4 \bar{A}_{ij} + \frac{1}{3}\psi^4 \bar{\gamma}_{ij} K \,, \quad \underline{\mathrm{tr}\, A_{ij} = 0}$$

Then evolve

$$\partial_t \bar{\gamma}_{ij} \qquad \partial_t (\psi^{-2})$$

$$\partial_t \bar{A}_{ij} \qquad \partial_t K \qquad \text{separately.}$$

All quantities remain bounded even <u>inside</u> <u>BH</u>!

<u>Final trick:</u>

Apply the above fix

Introduce $\bar{\Gamma}^i = \bar{\gamma}^{jk} \bar{\Gamma}^i_{jk} = -\partial_j \bar{\gamma}^{ij}$

Evolve $\partial_t \bar{\Gamma}^i = \ldots\ldots$

$$\bar{R}_{ij} \supset \bar{\gamma}_{k(i} \partial_{j)} \bar{\Gamma}^k + \bar{\Gamma}^k \bar{\Gamma}_{(ij)k}$$

$$+ \gamma^{lm} \partial_m \partial_n \bar{\gamma}_{ij}$$

proper wave equation

Bonus:

Imagine $\nabla \cdot B = 0$

$$\partial_t \vec{B} + \nabla \times E + \nabla \Phi = 0$$

$$\partial_t \Phi + \nabla \cdot B = 0$$

$$\rightarrow \quad \partial_t (\nabla \cdot B) + \Delta \Phi = 0$$
$$\partial_t \Phi + \nabla \cdot B = 0$$

$$\Leftrightarrow \quad \partial_t^2 (\nabla \cdot B) - \Delta(\nabla \cdot B) = 0$$

constraint violations propagate!

Need to do the same for BSSN!

$$R_{ab} + \boxed{\nabla_a Z_b + \nabla_b Z_a - k_1 (n_a Z_b + n_b Z_a - (1+k_2)\, g_{ab} n_c Z^c)}$$
$$= 8\pi \left(T_{ab} - \tfrac{1}{2} g_{ab} T\right)$$

constraint damping sector

$$\Theta = -n_\mu Z^\mu$$

$$\Rightarrow (\partial_t - \beta^l \partial_l)\, \Theta = \alpha H_0 - \alpha K \Theta$$

$$\partial_t H_i = \partial_t \left(\Theta D_i \ln \alpha - D_i \Theta \right)$$

$$\Rightarrow \partial_t H_i \sim D_i \partial_t \Theta \sim -D_i H_0$$

$$(\partial_t - \beta^l \partial_l) \tilde{\gamma}_{ij} \hat{=} -2 \alpha \bar{A}_{ij} + \dots$$

$$(\partial_t - \beta^l \partial_l) \tilde{A}_{ij} \hat{=} \alpha W^2 (R_{ij}^{TF} - 8\pi S_{ij}^{TF}) + D_i D_j \alpha + \dots$$

$$(\partial_t - \beta^l \partial_l) \varphi^{-2} = \frac{W}{3} \left(\hat{K} + 2\Theta \right) + \dots$$

$$(\partial_t - \beta^l \partial_l) \hat{K} = \bar{A}_{ij} \bar{A}^{ij} + K^2 + 4\pi \alpha (n_\mu n_\nu T^{\mu\nu} + S_k^k) + \dots$$

$$(\partial_t - \beta^l \partial_l) \hat{\Gamma}^i \hat{=} \Gamma^i_{jk} A^{jk} - 8\pi \gamma^{ik} j_k - 2\alpha \kappa_1 (\hat{\Gamma}^i - \hat{\Gamma}^i_\alpha) + \dots$$

Effectively this imposes the momentum constraint.

$$(\partial_t - \beta^l \partial_l) \Theta = \frac{1}{2} \alpha H_0 - \alpha \kappa_1 (2 + \kappa_2) \Theta$$

imposes H_0

Bonus 2, BH initial data

$$R + K^2 - K_{ij}K^{ij} = 0 \quad (1)$$

$$D_j(K^{ij} - \gamma^{ij}K) = 0 \quad (2)$$

conformal flatness: $g_{\mu\nu} = \psi^4 \eta_{\mu\nu}$, (Letters and CW)

maximal slicing: $K = 0$

$$\Rightarrow \boxed{D_i K^{ij} = 0} \quad (2)$$

$$\boxed{\Delta\psi + \frac{1}{8} K_{ab}K^{ab}\psi^{-7} = 0} \quad (1)$$

PN momenta

$$K^{ab}_{PS} = \frac{3}{2r^2}\left(P^a n^b + P^b n^a - (g^{ab} - n^a n^b) P^c n_c\right)$$

$$+ \frac{3}{r^3}\left(\varepsilon^{acd} S_c n_d n^b + \varepsilon^{bcd} S_c n_d n^a\right)$$

PN Spin

Now pick K^{ab}_{PS} for each BH

Choose $\psi = 1 + \sum \frac{m_i}{2|r - r_i|} + u$

Solve for ψ_4

Finding a BBH, when we know what BHs we have!

Riemann tensor has Ricci as trace!
But trace free part is still there

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - (\text{Ricci}\ g)$$
$$- (g g) R$$

10 components

$\psi_0, \ldots, \psi_4$ 5 complex scalars.

BH

null vectors

- l, k radially in, outgoing
  real null
- $l \cdot k = 1 = m \cdot \bar{m}$

real.

$$\Rightarrow \psi_4 = -C_{abcd}\, k^a \bar{m}^b k^c \bar{m}^d$$

outgoing radial null

Remember : $^{(4)}R_{abcd} = \frac{1}{2}(\partial_a\partial_d h_{bc} + \partial_b\partial_c h_{ad} - \partial_b\partial_d h_{ac} - \partial_a\partial_c h_{bd})$

Spherical symmetry:

$$^{(4)}R_{t\hat{\vartheta}t\vartheta} = -\frac{1}{2}\ddot{h}_+$$

$$\Rightarrow \boxed{\psi_4 = \ddot{h}_+ - i\ddot{h}_\times}$$