Why is PN necessary?

The gravitational waves emitted by binary systems have the following general form:

$$h(t) = \underset{\text{Amplitude}}{A(t)} \cos[\underset{\text{Phase.}}{\Phi(t)}]$$

~~where~~ Amplitude Phase.

$$\Phi(t) = \int^t dt' \omega_{gw}(t') = \int^t dt' \, 2\pi f_{gw}(t')$$

varies with time

$$\frac{d\Phi}{dt} = 2\pi f_{gw} \quad \text{— (1)}$$

To relate this relation to a binary velocity (assuming a circular orb we know that

$$V = R\omega_{orbit}$$

Using Kepler's law, $m = \omega_{orbit}^2 R^3$, $m \equiv m_1 + m_2$,

$$v = (m\omega_{orbit})^{1/3}$$

$$= (2m\pi f_{orbit})^{1/3}$$

$$= (\pi m f_{gw})^{1/3}$$

assuming quadrupole radiation $\omega_{gw} = 2\omega_{orbit}$.

$$\therefore \frac{d\Phi}{dt} = \frac{2v^3}{m} \quad \text{— (2)}$$

How does Φ evolve with v? Use chain rule

$$\left(\frac{d\Phi}{dv}\right)\underline{\underline{\left(\frac{dv}{dt}\right)}} = \frac{2v^3}{m}. \quad \text{— (3)}$$

$\frac{dv}{dt}$ can be obtained by using energy-balance argument, in which case we obtain

$$P = -\frac{dE}{dt} \quad \text{(conservation of energy)} \quad \text{— (4)}$$

where P is the power emitted (Einstein's quadrupole formula at leading order)

E = binding energy of the orbit.

$$\therefore P = -\frac{dE}{dt} = -\left(\frac{dE}{dv}\right)\left(\frac{dv}{dt}\right)$$

$$\Rightarrow \frac{dv}{dt} = -\frac{P}{dE/dv}$$

Substituting this relation to (3),

$$\frac{d\Phi}{dv} = -\frac{2v^3}{m}\cancel{\frac{P(v)}{dE/dv}}\left(\frac{dE/dv}{P(v)}\right)$$

$$\Phi(v) = \Phi_0 - \frac{2}{m}\int^v dv' \, v'^3 \boxed{\frac{dE(v')/dv'}{P(v')}}$$

$$t(v) = t_0 - \int^v dv' \boxed{\frac{dE(v')/dv'}{P(v')}}$$

conservative

Solve these to obtain the time-domain waveform

dissipative

Key Point: to solve for $\Phi(v)$ up to some PN order, we need to solve <u>both</u> $E(v)$ and $P(v)$ up to that same order.

How high a PN should we go to?

Consider the leading Newtonian case,

$$E = -\frac{m_1 m_2}{R} + \frac{1}{2}\left(\frac{m_1 m_2}{m}\right) v^2$$

From virial theorem (after orbit averaging),

$$\frac{m}{R} = v^2$$

$$E = -\frac{1}{2}\eta m v^2, \quad \text{where } \underset{\text{symmetric mass ratio}}{\eta} \equiv \frac{m_1 m_2}{(m_1+m_2)^2}, \quad m \equiv m_1 + m_2$$

From Einstein's quadrupole formula,

$$P = \frac{32}{5}\eta^2 v^{10}$$

$$\Phi(v) = \Phi_0 - \frac{2}{m}\int^v dv' \left[\frac{-5\eta m v'}{32\eta^2 v'^{10}}\right] v'^3$$

$$= \Phi_0 + \underbrace{\frac{5}{16}\frac{1}{\eta}\int^v dv' \frac{1}{v'^6}}$$

$$\Phi(v) - \Phi_0 = -\frac{1}{16}\boxed{\frac{1}{\eta}\frac{1}{v^5}}$$

Scales inversely with v^5!

Using $v = (\pi m f_{gw})^{1/3}$, $\quad \Phi(v) - \Phi_0 = -\frac{1}{16}\left(\frac{m^2}{m_1 m_2}\right)\frac{1}{(\pi f_{gw} m)^{5/3}}$

$$= -\frac{1}{16}\frac{m^{1/3}}{m_1 m_2}\frac{1}{f_{gw}^{5/3}} \propto \frac{1}{\mathcal{M}_c^{5/3}}\frac{1}{f_{gw}^{5/3}}$$

where $\mathcal{M}_c \equiv \frac{(m_1 m_2)}{m}$

$$\Phi(v) \propto \frac{1}{\mathcal{M}_c^{5/3} f_{gw}^{5/3}} \Big[1 + \# f^{2/3} + \# f^{4/3} + \ldots\ldots \Big]$$

$$\underset{(1\text{PN})}{\sim v^2} \qquad \underset{(2\text{PN})}{\sim v^4} \qquad \ldots \ldots$$

Key point: ① At leading order, we measure $\mathcal{M}_c$ but not the individual mass components

② To achieve n^{th} PN, we need BOTH $E(v)$ and $P(v)$ up to n^{th} PN

~~② For terms that scale as~~

③ The phase formally diverges below 2.5PN order

$$\Phi(v) \propto \frac{1}{\eta v^5} \Big[1 + v^2 + v^4 + v^6 + v^8 + \ldots\ldots \Big]$$

formally diverges as $v \to 0$

$\therefore$ becomes increasingly important as you observe many GW cycles.

become important as the binary approaches merger.

$\therefore$ Naively, to achieve $\lesssim O(1)$ radian in phase accuracy, you would therefore conclude that one can stop the PN computations at 2.5PN order.

However, turns out the coefficients of these terms are larger than order unity.

$\longrightarrow$ Show slides (Early days estimation: requires up to 3.5PN order)

[arXiv: 0907.0700]

The phase contains ~~many~~ information the nature of the binary constituents.

$$\Phi = \underbrace{\Phi_{pp}}_{\text{point particle}} + \underbrace{\Phi_s}_{\text{spin-dependent terms.}} + \Phi_T \leftarrow \text{tidal terms} \quad \text{"love numbers"}$$

→ ~~But~~ Highly accurate PN calculations are needed not only to be able to detect these signals to begin, but allows for accurate parameter estimation.

— Before proceeding

~~h(t)~~ what about amplitude? At leading order

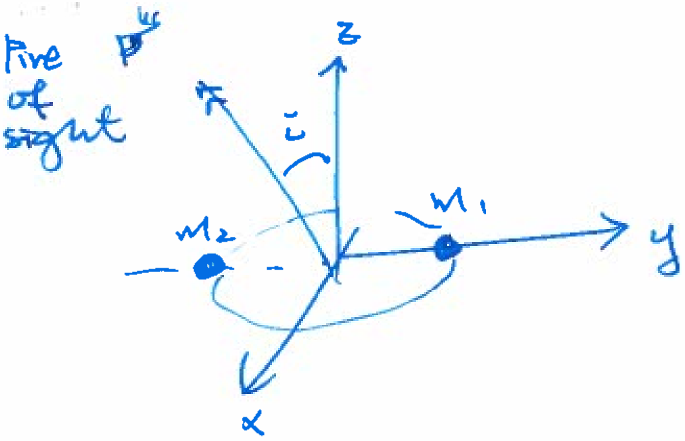
$$h_+(t) = \frac{4}{d} M_c^{5/3} (\pi f_{gw})^{2/3} \left[\frac{1+\cos^2 i}{2}\right] \cos[\Phi(t)]$$

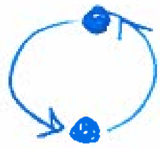
$$h_\times(t) = \frac{4}{d} M_c^{5/3} (\pi f_{gw})^{2/3} \cos i \; \sin[\Phi(t)].$$

— Lev described the scaling

- PN corrections here as not as important, simply normalizes amplitude / distance.

- By measuring M_c in the phase, we break the degeneracy between M_c and d, though d is still very degenerate with the inclination angle i





$i=0$, both h_+ and $h_\times$ contribute to the same magnitude
$i=\pi/2$, we are blind to one of the polarizations.

PN expansion

There is a whole zoo of techniques developed to compute these relativistic effects of binary systems

- post-Newtonian expansion, v
- post-Minkowskian expansion, G

} Minkowski background metric.

  - Classic metric approach ← Sketch the idea behind this approach.
  - EFT approach
  - Scattering amplitudes
  - ADM Hamiltonian formalism.

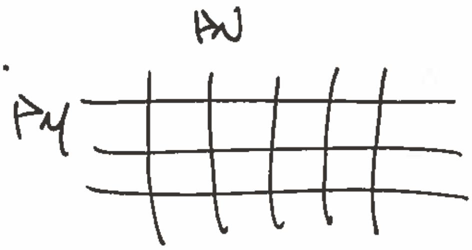
- Black-hole perturbation theory, η → Kerr background metric
  - Teukolsky equation

All these expansions are ~~ex~~ complement one another because

1) Virial theorem: $\frac{GM}{R} = v^2$ ⇒ useful power-counting tool

∴ This is a double series.

→ Refer to ~~these~~ PN-PM diagram.
[arXiv: 1901.04424]



2) In fact, this is a triple series, because the coefficients depend on η

From last week,

- geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

- linearized Einstein equation:

$$\underset{\substack{\uparrow \\ \text{Flat metric.}}}{\eta^{\alpha\beta}}\partial_\alpha\partial_\beta \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \quad \partial_\mu \bar{h}^{\mu\nu} = 0 \quad \text{Lorenz / harmonic gauge}$$

$$\Downarrow$$

$$\partial_\mu T^{\mu\nu} = 0$$

EM conservation is naturally enforced.

$$\cancel{\bar{h}_{\mu\nu} = \frac{4G}{c^4}\int d^3y \frac{T_{\mu\nu}(t-|x-y|, y)}{|x-y|}}$$

Post-Newtonian Expansion, $V^\alpha \equiv \frac{dx^\alpha}{d\tau} = \left(\frac{dx}{dt}, V^i\right)$

implies that $\boxed{|v^i| \ll 1, \ \tau \simeq t}$, $\frac{\partial}{\partial t} = \underbrace{\frac{\partial x^i}{\partial t}\frac{\partial}{\partial x^i} = O(v)\frac{\partial}{\partial x^i}}_{\text{time derivatives are suppressed.}}$

$\therefore$ Geodesic equation:

$$\frac{d^2x^i}{dt^2} + \Gamma^i{}_{00}\underbrace{\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}}_{\simeq 1} \simeq 0$$

$$\frac{d^2x^i}{dt^2} = \partial^i\left(\frac{h_{00}}{2}\right) \equiv -\partial^i U, \quad U \equiv -\frac{h_{00}}{2}.$$

~~where~~ If $U = -\frac{M}{R}$, $\frac{d^2x^i}{dt^2} = -\frac{M}{R^2}$ Newtonian force & inverse square law

Einstein's equation:

$$-\frac{\partial^2 \bar{h}_{00}}{\partial t^2} + \nabla^2 \bar{h}_{00} = -\frac{16\pi G}{c^4}T_{00} \quad T_{00} \equiv \rho \text{ (energy density)}$$

~~Similarly.~~ $\nabla^2 U = 8\pi G\rho$ (Poisson equation)

For a binary system, (Newtonian Order).

$$a^i \equiv \frac{d\xi^i}{dt^2} = -\frac{m}{R^2}\hat{R}^i$$

$$E = \eta m \left(\frac{1}{2}v^2 - \frac{m}{R}\right)$$

$$L^i_N = \eta m \,\epsilon^{ijk} R_j v_k .$$

In what follows:

$$a^i = a^i_N + a^i_{1PN} + a^i_{2PN} + \ldots$$

$$E = E_N + E_{1PN} + E_{2PN} + \ldots$$

$$L^i = L^i_N + L^i_{1PN} + L^i_{2PN} + \ldots$$

Post-Newtonian: higher v-expansion in the metric.

$$g_{00} = -1 + \underbrace{g_{00}^{(2)}}_{\text{Newtonian}} + \underbrace{g_{00}^{(4)} + \ldots}_{\text{1PN}}$$

$$g_{0i} = g_{0i}^{(3)} + \ldots$$

$$g_{ij} = \delta_{ij} + g_{ij}^{(2)} + \ldots$$

Newtonian   1PN   - - - - -

$$T_{00} = T_{00}^{(0)} + T_{00}^{(2)} + \ldots \quad , \quad T_{00} \sim \rho c^2$$

$$T_{0i} = T_{0i}^{(1)} + \ldots \quad , \quad T_{0i} \sim \rho c v^i$$

$$T_{ij} = T_{ij} + \ldots \quad , \quad T_{ij} \sim \rho v^i v^j$$

$$\nabla^2 \left[{}^{(2)}g_{ij} \right] = -\frac{8\pi G}{c^4} \delta_{ij} T^{00}$$

$$\nabla^2 \left[{}^{(3)}g_{0i} \right] = \frac{16\pi G}{c^4} T^{0i}$$

$$\nabla^2 \left[{}^{(4)}g_{00} \right] = \partial_0^2 \left[{}^{(2)}g_{00} \right] + {}^{(2)}g_{ij}\, \partial_i \partial_j \left[{}^{(2)}g_{00} \right] - \partial_i \left[{}^{(2)}g_{00} \right] \partial_i \left[{}^{(2)}g_{00} \right]$$
$$- \frac{8\pi G}{c^4} \left({}^{(2)}T^{00} + {}^{(2)}T^{ii} - 2\, {}^{(2)}g_{00}\, {}^{(0)}T^{00} \right)$$

(BORING.)

for a binary system: Lorentz-Droste-Einstein-Infeld-Hoffmann

$$a^i_{1PN} = -\frac{m}{R^2}\left\{\left[(1+3\eta)v^2 - (4+2\eta)\frac{m}{R} - \frac{3}{2}\eta \dot{R}^2\right]\hat{R}^i - (4-2\eta)\dot{R}v^i\right\}$$

$c = 1\,G$

· Virial theorem: $v v^2$ suppressed.

→ Presence of a tangential force (perihelion precession) cf. mercury around sun

$f^2 \sim \frac{m}{R^3}$

$\sim \frac{\cancel{m} v^6}{m^2}$

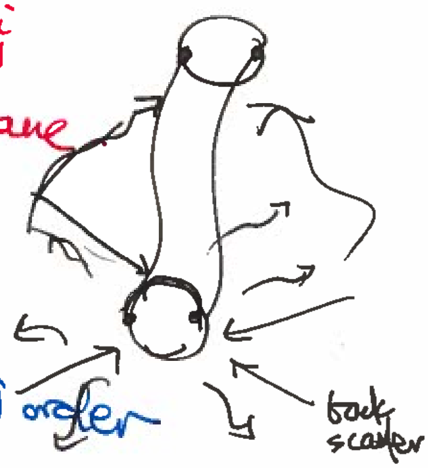
$f \sim \frac{v^3}{m}$

$$E_{1PN} = \eta m\left\{\frac{3}{8}(1-3\eta)v^4 + \frac{m}{2R}\left((3+\eta)v^2 + \eta\dot{R}^2 + \frac{m}{R}\right)\right\}$$

$$L^i_{1PN} = L^i_N\left[\frac{1}{2}(1-3\eta)v^2 + (3+\eta)\frac{m}{R}\right]$$

→ η starts to appear explicitly (instead as an overall scale)
∴ can measure mass ratio at 1PN onwards

→ L^i_{1PN} is aligned with Newtonian L^i_N
∴ orbit still occurs on a fixed plane



⋮

→ Qualitatively unchanged up until 4PN order for point particles without spin

~~⇒ Phase for pp only appears in even powers of v because it doesn~~'t

At 4PN, TAIL EFFECT $\ln(v)$

Limitation of PN expansion:

$$\left[-\frac{\partial^2}{\partial t^2}+\nabla^2\right]\bar h_{\mu\nu}(\underbrace{t-r/c}_{\text{retarded } \cancel{\text{effect}} \text{ time.}})$$

By treating ∂_t and ∇ on an equal footing,

$$\partial_t \bar h_{\mu\nu} = v^i \partial_i h_{\mu\nu} \ll \partial_i h_{\mu\nu}$$

Retarded effects are small corrections to instantaneous effects.

small $t-r$, $\cancel{\partial_t \bar h_{\mu\nu}}$ $\bar h_{\mu\nu}(t-r) = \bar h_{\mu\nu}(t) - r\partial_0 \bar h_{\mu\nu} + \dots\dots$

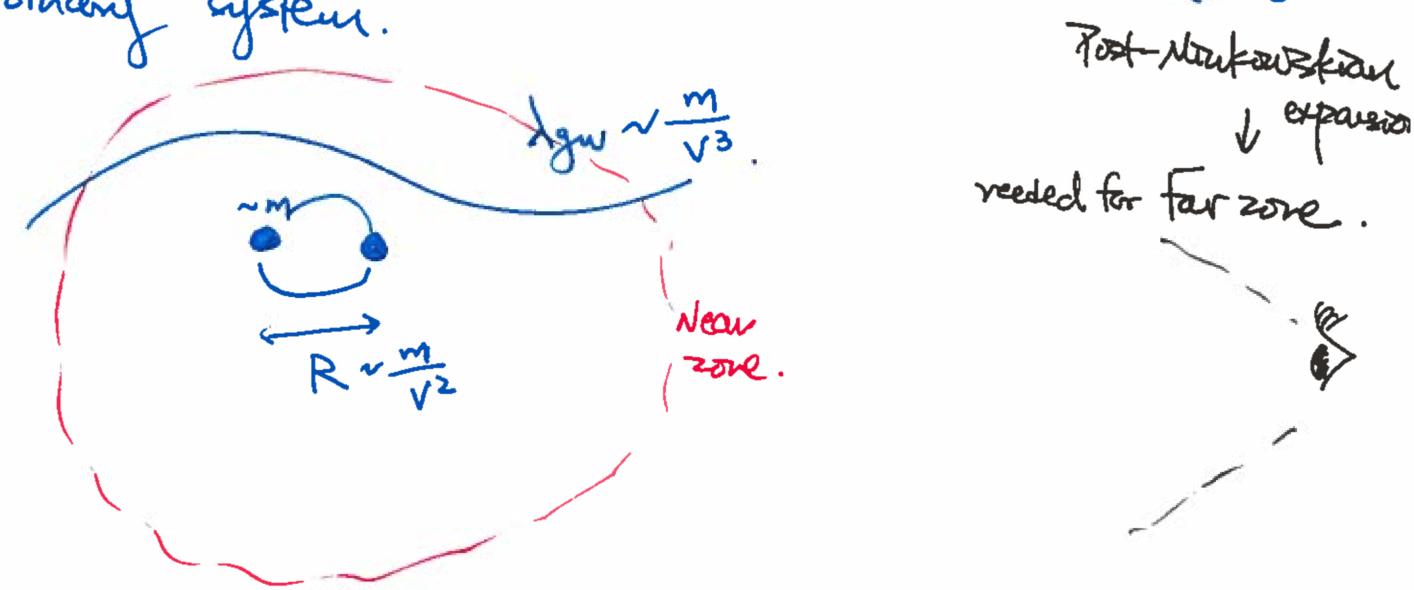
Since $\partial_0 \bar h_{\mu\nu} \sim \dfrac{\bar h_{\mu\nu}}{\lambda_{gw}}$.

$$r\partial_0 \bar h_{\mu\nu} \sim \left(\frac{r}{\lambda_{gw}}\right)\bar h_{\mu\nu}$$

$\downarrow$

diverges as $r \gg \lambda_{gw}$.

$\therefore$ PN expansion is only valid in the "near zone" of the binary system.



PM expansion

In the classic approach to computing PM expansion, the Einstein equation is recasted into the Landau-Lifshitz form.

Introduce the "gothic metric",

$$\mathfrak{h}^{\mu\nu} \equiv \eta^{\mu\nu} - \sqrt{-g}\, g^{\mu\nu}$$

where $\mathfrak{h}_{\mu\nu}$ is NOT assumed to be small. (exact definition)

In the weak field limit, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $\mathfrak{h}^{\mu\nu} = h^{\mu\nu} + \ldots$

Dedonder gauge $\partial_\mu \mathfrak{h}^{\mu\nu} = 0$

the full Einstein equation $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ can be rewritten as

$$\Box_\eta \mathfrak{h}^{\mu\nu} = -\frac{16\pi G}{c^4} \tau^{\mu\nu}$$

where $\Box_\eta = -\partial_t^2 + \nabla^2$ (flat d'Alembertian).

and $\tau^{\mu\nu} \equiv \underbrace{(-g) T^{\mu\nu}}_{\text{EM tensor of matter}} + \underbrace{\frac{c^4}{16\pi G} \Lambda^{\mu\nu}}_{\text{"EM tensor of vacuum"}}$

In PM expansion,

$$\sqrt{-g}\, g^{\mu\nu} = \eta^{\mu\nu} - G h_1^{\mu\nu} - G^2 h_2^{\mu\nu} + \cdots$$

$$\Lambda^{\mu\nu} = \Lambda_2^{\mu\nu}[h,h] + \Lambda_3^{\mu\nu}[h,h,h] + \Lambda_4^{\mu\nu}[h,h,h,h] + \cdots$$

Solve these equations order by order.

In ~~vacuum~~ Far region, $T_{\mu\nu}=0$ (vacuum)

$$\Box h_1^{\mu\nu} = 0 \quad \text{— linearized EE.}$$

$$\Box h_2^{\mu\nu} = \Lambda_2^{\mu\nu}[h_1,h_1]$$

$$\Box h_3^{\mu\nu} = \Lambda_3^{\mu\nu}[h_1,h_1] + \Lambda_2^{\mu\nu}[h_1,h_2] + \Lambda_2^{\mu\nu}[h_2,h_1]$$

$$\vdots$$

$$h_1^{\mu\nu} = \sum_{\ell=0}^{\infty} \partial_L \Big[\underbrace{\tfrac{1}{r} K_L^{\mu\alpha}(t-r)}_{\text{retarded wave solution.}} \Big] \quad \Big\} \text{ wave-zone solution.}$$

In the near region, $T_{\mu\nu} \neq 0$ $\rightarrow$ Expand them in the basis of spherical harmonics as well

Technical details : Poisson, Will

Maggiore

$$\Lambda^{\mu\nu} = \frac{16\pi G}{c^4}(-g)t^{\mu\nu}_{LL} + \left(\partial_\beta h^{\alpha\nu,\mu\alpha}\partial_\alpha h^{\beta\nu} - h^{\alpha\beta}\partial_\alpha\partial_\beta h^{\mu\nu}\right)$$

and $t^{\mu\nu}_{LL}$ is the Landau Lifshitz tensor.

$$\frac{16\pi G}{c^4} t^{\mu\nu}_{LL} = \cancel{g_{\lambda\alpha} g_{\beta\rho}}\cdot \text{ horrible expression}$$

that scales, quadratically in $h^{\mu\alpha}$ dominantly as

→ Key Point:

1. $\Box_\eta$ = flatspacetime d'Alembertian
   ∴ can solve using the standard Green's function method
2. $\Lambda^{\mu\nu}$ only depends on the metric & encodes non-linearity of gravity & the metric.

At leading order in G, Λ vanishes

$$\Box_\eta \bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4}T^{\mu\nu} \rightarrow \text{linearized Einstein equation.}$$

By matching the dissipative effect in the far region with the conservative dynamics in the near region, we can find the radiative reaction force

$$a^i_{RR} = \frac{8\eta}{5}\frac{m^2}{R^3}\left[\left(3v^2+\frac{17}{3}\frac{m}{R}\right)\dot{R}\hat{R}^i - \left(v^2-\frac{3m}{R}\right)v^i\right]$$

$$\sim \frac{m}{R^2}O(v^5) \qquad \text{Burke-Thorne}$$

$\therefore$ Radiative reaction force is 2.5PN suppressed compared to the leading Newtonian force

$\Rightarrow$ ** E, L^i are no longer conserved at 2.5PN and higher.

Simple way of understanding this 2.5PN effect from power counting:

$$P = \frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle \sim \eta^2 m^2 R^4 \Omega^6$$

$$\sim \eta^2 m^2 \left(\frac{m}{\Omega^2}\right)^{4/3}\Omega^6$$

$$\sim \eta^2 (m\Omega)^{10/3}, \quad \text{where } v \sim (m\Omega)^{1/3}$$

$$\sim \eta^2 v^{10}$$

Since $P = F_{RR} v$, $F_{RR} = \eta m a_{RR}$

$$\cancel{F_{RR} = \underbrace{\eta m}_{\text{reduced mass}} a^i_{RR} \sim \eta^2 (m\Omega)^{10/3}\frac{1}{v}}$$

$$\cancel{a^i_{RR} \sim \eta m^{2/3}\Omega^{5/3}} \sim v^5$$

$$a_{RR} = \frac{F_{RR}}{\eta m} = \frac{(P/v)}{\eta m}$$

$$\approx \frac{\eta^2 v^{9}}{\eta m}$$

$$\sim \frac{\eta}{m} v^{9}$$

Virial theorem, $v^2 \sim \frac{m}{R}$

$$\therefore a_{RR} \sim \frac{\eta}{m}\left(\frac{m}{R}\right)^2 v^5$$

$$\sim \underbrace{\left(\frac{\eta m}{R^2}\right)}_{\text{Newton}} v^5$$

Key Message: Dissipative effects are always 2.5PN suppressed compared to conservative effects.

$$\Phi(v) = \Phi_0 - \int^{v} dv'\, v'^3 \frac{dE(v')/dv'}{P(v')}.$$

$$\cancel{= \Phi_0 - \int^{v} dv'\, v'^3 \frac{d}{dv'}\left(E_N + E_{1PN} + \dots\right)}$$

PN counting, can be confusing language.

$$E(v) = -\frac{1}{2}\eta m v^2 \Big(1 + \underset{v^2}{1PN} + \underset{v^4}{2PN} + \dots \Big)$$

$$P(v) = \underbrace{\frac{32}{5}\eta^2 v^{10}}_{\text{even though } v^5\text{-suppressed compared to leading conservative}} \Big(1 + 1PN + 2PN + \dots \Big)$$