Lecture 1: The basics of general relativity

December 12, 2020

This is my attempt at a 2-hour-long overview of general relativity (GR). I am focusing only on topics relevant to future discussions of gravitational waves (GW).

Literature

I used primarily Hawking & Ellis and Landau & Lifshitz (vol 2). I like these two books because they barely have any approach in common. It is nice to see things from multiple angles.

Topics covered

We start with general remarks about GR and its origin. then we dive into manifolds. In general relativity space-time is treated as a four-dimentional maniold. We introduce manifolds and the structures living on them. Then we discuss how to operate with these structures. We define Riemann tensor and its physical meaning. We write Einstein's equation. We take a limit to obtain Newtonian gravity. The purpose of this is to show how $g_{\mu\nu}$ looks in Newtonian case and to demonstrate that we can indeed absorb the gravitational force into $g_{\mu\nu}$. Then we derive gravitational waves and calculate binary-star inspiral. The last two scanned pages are taken from lectures by Chris Hirata.

Hawking & Ellis
Landau & Lifshitz

Lecture 1

GR refresher

Gravity acts on all objects in the same way irrespect of their mass.
This is similar to non-inertial frames.

gravity $\longleftrightarrow$ non-inertial frames

In inertial Galilean frame interval ds^2:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

Lets move to a rotating frame:

$$x = x'\cos\Omega t - y'\sin\Omega t$$
$$y = x'\sin\Omega t + y'\cos\Omega t$$
$$z = z'$$

$$ds^2 = -(c^2 - \Omega^2(x'^2+y'^2))dt + dx'^2 + dy'^2 + dz'^2$$
$$- 2\Omega y'dx'dt + 2\Omega x'dy'dt$$
$$= g_{\mu\nu}\,dx^\mu dx^\nu$$

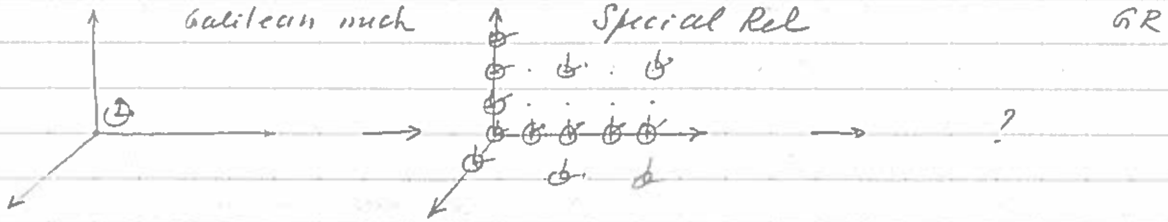
$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix} \longrightarrow \text{more general form}$$

Gravity can be accounted for by modifying $g_{\mu\nu}$, i.e. the geometry of space and time

Warnings:

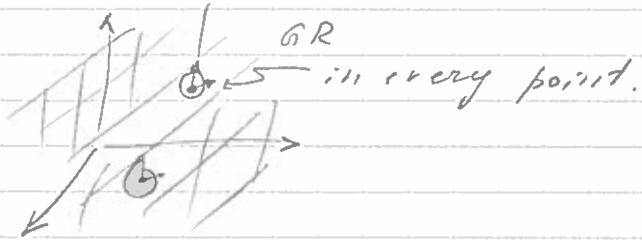
- True grav. field can be locally cancelled with a choice of ref. frame, but <u>NOT</u> globally
- At $r \to \infty$ grav. field $\to 0$, i.e. $g_{\mu\nu} \to \mathrm{diag}(-1,1,..)$

Reference frames:



There is no relative rest in GR
Look at a variable grav field. Metric changes all the time and everywhere. => distances change all the time..

All space in GR is filled with bodies and clocks attached to them



Ref. frames in GR isn't equivalent
Phys. laws have different forms in each frame.

We should write all phys. laws in covariant form, i.e the form which is the same in every ref frame.

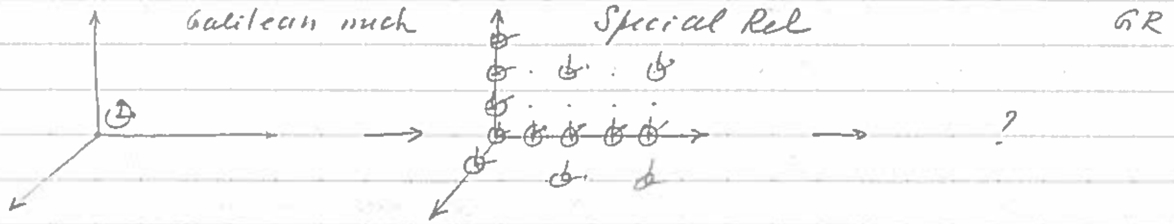
Main idea of GR

matter $\longrightarrow$ geometry

$T^{\mu\nu}$ = tensor with the same prop. describing geometry of space-time

energy-mom tensor

Reference frames:



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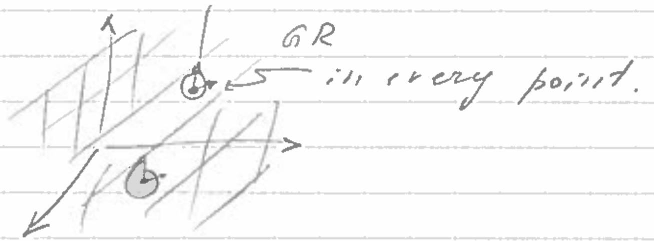

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Vector, tensors etc in manifolds (M)

[Paracompact, connected C^∞ Hausdorff manifold without boundary] — differentiable etc

Vector:

Define a curve $\lambda(t)$ in M. And function $f(\lambda(t))$

Vector (covariant vector) $(\partial/\partial t)_\lambda |_{t_0}$ tangent to $\lambda(t)$ at $\lambda(t_0)$ is the operator

$$f(\lambda(t_0)) \longrightarrow \left(\frac{\partial f}{\partial t}\right)_\lambda \Big|_{t_0} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \{ f(\lambda(t_0+\Delta t)) - f(\lambda(t_0)) \}$$

If x^μ are coordinates in the neighbourhood of $p = \lambda(t_0)$

$$\left(\frac{\partial f}{\partial t}\right)_\lambda \Big|_{t_0} = \frac{\partial f}{\partial x^\mu}\Big|_{\lambda(t_0)} \frac{dx^\mu(\lambda(t))}{dt}\Big|_{t=t_0} = \frac{dx^\mu}{dt} \frac{\partial f}{\partial x^\mu}\Big|_{\lambda(t_0)}$$

Any tangent vector at p can be expressed as

$$V = V^\mu \left(\frac{\partial}{\partial x^\mu}\right)\Big|_p , \quad V^\mu \text{ are numbers}$$

Space of all tangent vectors to M at p is $T_p(M)$

We can introduce basis

$\{e_\mu\}$ - set of lin independent vectors at p; number of vectors is equal to dim of the manifold.

Any vector $V = V^\mu e_\mu$

One-form (covariant vector) ω is a Real valued function on space T_p of vectors at p. If X is a vector at p, then ω maps X into a number $\langle \omega, X \rangle$

All one-forms at p form space T_p^*

Each function f on M defines a one-form df at p by rule that for each X

$$\langle df, X \rangle = f_X = \frac{\partial f}{\partial x^\mu} X^\mu$$

We define basis of one forms using basis of vectors e_μ

$$\langle e^\mu, e_\nu \rangle = \delta^\mu{}_\nu$$

As $df = \dfrac{\partial f}{\partial x^\mu} dx^\mu$ we can define basis using local coordinates

Set $\{dx^\mu\}$ - basis on 1-forms

$\left\{\dfrac{\partial}{\partial x^\mu}\right\}$ basis of vectors dual to $\{dx^\mu\}$

$$V = V^\mu \frac{\partial}{\partial x^\mu} = \underbrace{V^\mu \frac{\partial x'^\nu}{\partial x^\mu}}_{= V'^\nu} \frac{\partial}{\partial x'^\nu}$$

$$W = w_\mu dx^\mu = \underbrace{w_\mu \frac{\partial x^\mu}{\partial x'^\nu}}_{= w'_\nu} dx'^\nu$$

Date

Tensors

Cartesian product $\Pi^s_r = \underbrace{T_p^* \times \dots T_p^*}_{r} \times \underbrace{T_p \times \dots \times T_p}_{s}$

We define tensors as objects acting on Π^s_r.
and returning a number

Space of all such tensors:

$$T^r_s = \underbrace{T_p \otimes \dots \otimes T_p}_{r} \otimes \underbrace{T_p^* \otimes \dots \otimes T_p^*}_{s}$$

Obvious that tensors will be transforming with coord transfor

$$T^{\mu_1' \dots \mu_r'}{}_{\nu_1' \dots \nu_s'} = T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \dots \frac{\partial x^{\mu_r'}}{\partial x^{\mu_r}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \dots \frac{\partial x^{\nu_s}}{\partial x^{\nu_s'}}$$

Date

Metric tensor

A metric tensor at point $p \in M$ is a symmetric tensor of type $(0,2)$. g at p assigns a "magnitude" $\sqrt{g(X,X)}$ to each $X \in T_p$ and "angles" $\frac{g(X,Y)}{\sqrt{|g(X,X) \cdot g(Y,Y)|}}$ between any $X, Y \in T_p$ such that $g(X,X) \cdot g(Y,Y) \neq 0$.

In coord basis
$\hat{g} = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$

The components
$g_{\mu\nu} = g(e_\mu, e_\nu) = g(e_\nu, e_\mu)$

$$g\, V X = g_{\mu\nu}\, dx^\mu \otimes dx^\nu \; V^\alpha \frac{\partial}{\partial x^\alpha} X^\beta \frac{\partial}{\partial x^\beta} = g_{\alpha\beta} X^\alpha X^\beta$$

From now we will use
$ds^2 = g_{\mu\nu} \underbrace{dx^\mu dx^\nu}_{\text{vector components}}$ - infinitesimal arc between $x^\mu \to x^\mu + dx^\mu$

$g(\cdot,\cdot)$ takes 2 vectors and sends them to a number
$g(V,\cdot)$ takes 1 vector and sends it to a number.

For each vector V (with comp V^μ) we can find a co-vector $\tilde{V}$ with component $V_\mu = g_{\mu\nu} V^\nu$

We'll drop $\tilde{V}$ and use g to rise and lower indexes in vectors and tensors!

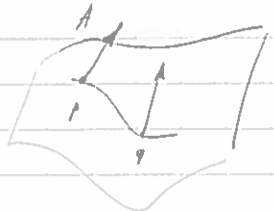
Covariant derivative. Christoffel symbols.

We discussed what lives on our manifolds and how they transform. Let's move toward physics.
We need a covariant derivative, i.e. an expression for a derivative useful in every ref. frame.

An ordinary one doesn't work

$$A_\mu = \frac{\partial x'^\nu}{\partial x^\mu} A'_\nu$$

$$dA_\mu = \frac{\partial x'^\nu}{\partial x^\mu} dA'_\nu + A'_\nu \frac{\partial^2 x'^\nu}{\partial x^\mu \partial x^\alpha} dx^\alpha$$



We need to move A(q) to p. (Its coord would change) and then take dA
We define
$$DA^\mu = dA^\mu - \delta A^\mu$$

δA^μ for $q \to p$ with $|p-q| \to 0$ can only be a function of A^ν and coord. The dependence must be linear

$$\delta A^\mu = -\Gamma^\mu_{\alpha\beta} A^\alpha dx^\beta$$

$\Gamma^\mu_{\alpha\beta}$ are Christoffel symbols

$$\Gamma_{\mu,\alpha\beta} = g_{\mu\gamma} \Gamma^\gamma_{\alpha\beta}$$

So we now write:

$$DA^\mu = \left(\frac{\partial A^\mu}{\partial x^\nu} + \Gamma^\mu_{\alpha\nu} A^\alpha\right) dx^\nu$$

$$DA_\mu = \left(\frac{\partial A_\mu}{\partial x^\nu} - \Gamma^\beta_{\mu\nu} A_\beta\right) dx^\nu$$

Properties of Christoffels:
$$\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$$

$$A^\mu_{;\nu} = \frac{\partial A^\mu}{\partial x^\nu} + \Gamma^\mu_{\alpha\nu} A^\alpha$$

$$A_{\mu;\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \Gamma^\alpha_{\mu\nu} A_\alpha$$

We can expand this notion to tensors

$$A_{\mu\nu;\alpha} = \frac{\partial A_{\mu\nu}}{\partial x^\alpha} - \Gamma^\beta_{\mu\alpha} A_{\beta\nu} - \Gamma^\beta_{\nu\alpha} A_{\beta\mu}$$
etc

To derive Christoffels we take covariant derivative of metric tensor

First we notice that $g_{\mu\nu;\alpha} = 0$
DA^μ is a vector $\Rightarrow$ $DA_\mu = g_{\mu\nu} DA^\nu$
but
$$A_\mu = g_{\mu\nu} A^\nu$$

$$\Rightarrow DA_\mu = D(g_{\mu\nu} A^\nu) = g_{\mu\nu} DA^\nu + \underbrace{A^\nu D g_{\mu\nu}}_{0}$$

By covariant differentiating $g_{\mu\nu}$ we get

$$\Gamma_{\mu,\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right)$$

Geodesics

Free particle motion in special relativity is described by

$$\frac{du^\mu}{d\tau}=0\quad,\quad u^\mu=\frac{dx^\mu}{d\tau}\qquad c^2d\tau^2=c^2dt^2-dx^2-dy^2-dz^2$$

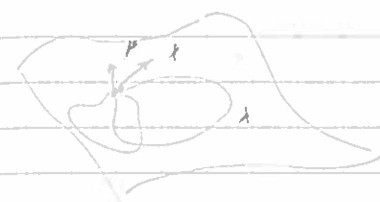
In curved space-time this equation would look like

$$\frac{Du^\mu}{d\tau}=0$$

$$\frac{du^\mu}{d\tau}+\Gamma^\mu_{\alpha\beta}u^\alpha\frac{dx^\beta}{d\tau}=0$$

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=0\quad - \text{ geodesics}$$

Riemann tensor



If we take a vector in p and a curve in M which starts and ends in p and parallel transfer X along λ, we'll end up with $X'\neq X$.

If we take λ' (a different curve) we'll end up with $X''\neq X'\neq X$

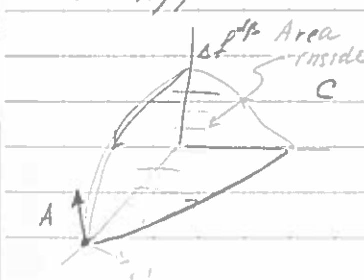
This is the consequence of the fact that covariant derivatives do not commute

Riemann tensor gives the measure of this non-communation.

1st Approach

$$A^\gamma{}_{;\beta\alpha}-A^\gamma{}_{;\alpha\beta}=R^\gamma{}_{\mu\alpha\beta}A^\mu$$

2nd Approach



Take vector A and drag it around contour C. The resulting change in A will be ΔA

$$\Delta A^\gamma=-\frac{1}{2}R^\gamma{}_{\mu\alpha\beta}A^\mu\Delta f^{\alpha\beta}$$

$$R^\gamma_{\mu\alpha\beta}=\frac{\partial\Gamma^\gamma_{\mu\beta}}{\partial x^\alpha}+\Gamma^\gamma_{\nu\alpha}\Gamma^\nu_{\mu\beta}-\frac{\partial\Gamma^\gamma_{\mu\alpha}}{\partial x^\beta}-\Gamma^\gamma_{\nu\beta}\Gamma^\nu_{\mu\alpha}$$

Properties of Riemann tensors:

$$R_{\mu\nu\alpha\beta}=g_{\mu\gamma}R^\gamma{}_{\nu\alpha\beta}$$

$$R_{\mu\nu\alpha\beta}=-R_{\nu\mu\alpha\beta}=-R_{\mu\nu\beta\alpha}$$

$$R_{\mu\nu\alpha\beta}=R_{\alpha\beta\mu\nu}$$

$$R_{\mu\nu\alpha\beta}+R_{\mu\beta\nu\alpha}+R_{\mu\alpha\beta\nu}=0$$

$$R^\gamma{}_{\mu\nu\alpha;\beta}+R^\gamma{}_{\mu\beta\nu;\alpha}+R^\gamma{}_{\mu\alpha\beta;\nu}=0$$

Dato

Einstein's equations:

$R^{\sigma}{}_{\mu\nu\alpha} \longrightarrow$ Ricci tensor $\longrightarrow$ Ricci scalar

$R_{\mu\alpha} = R^{\sigma}{}_{\mu\sigma\alpha}$ $\quad R = g^{\mu\nu} R_{\mu\nu}$

Matter $\sim$ Geometry

$T_{\mu\nu}$ $\quad R_{\mu\nu}, g^{\mu\nu}$

$$\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

or

$$R_{\mu\nu} = \underset{\frac{8\pi G}{c^4}}{8\pi} \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) + \Lambda g_{\mu\nu}$$

Dato

Newtonean limit

Non-relativistic particle in grav. field:

$$L = -mc^2 + \frac{mv^2}{2} - m\varphi$$

$$S = \int L dt = -mc \int \left(c - \frac{v^2}{2c} + \frac{\varphi}{c}\right) dt$$

$$dS = -\left(c - \frac{v^2}{2c} + \frac{\varphi}{c}\right) dt$$

Calculating ds^2 and taking limit $c \to \infty$

$$ds^2 = -(c^2 + 2\varphi) dt^2 + dr^2 \qquad \frac{dr^2}{dt^2} dt^2 = v^2 dt^2$$

$$g_{00} = 1 + \frac{2\varphi}{c^2}$$

Energy momentum tensor of slowly moving particles producing weak grav. field

$$T_{\mu\nu} = \rho c^2 u_\mu u_\nu$$

4-velocity u_μ is such (in this limit) that

$u_0 = 1$

$u_a = 0$

$$T^0{}_0 = \rho c^2$$

$$R^\mu{}_\nu = \frac{8\pi G}{c^4}\left(T^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu T\right)$$

$$R^0{}_0 = \frac{4\pi G}{c^2} M$$

$$R^0{}_0 = R_{00} = \frac{\partial \Gamma^\alpha{}_{00}}{\partial x^\alpha} \simeq \frac{\partial}{\partial x^\alpha}\left(-\frac{1}{2} g^{\alpha\beta} \cdot \frac{\partial g_{00}}{\partial x^\beta}\right) = \frac{\partial}{\partial x^\alpha}\left(\frac{1}{c^2} \frac{\partial \varphi}{\partial x^\alpha}\right) = \frac{1}{c^2}\Delta\varphi$$

$$\Delta\varphi = 4\pi G \rho$$

$$\varphi = -\frac{GM}{R}$$

If we actually solve it carefully:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 R}\right) dt^2 + \left(1 + \frac{2GM}{c^2 R}\right)[dx^2 + dy^2 + dz^2]$$

Gravitational Waves $\quad g^{\mu\nu}$ $\quad$ weak perturbation

Linearized gravity: $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$

Note: $g^{(0)}_{\mu\nu}$ rises and lowers indexes.

If we introduce an additional perturbation

$x'^\mu = x^\mu + \xi^\mu \quad$ with $\xi^\mu \ll 1$

Then $g_{\mu'\nu'}(x') = \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} g_{\alpha\beta}(x)$

$$\simeq (\delta^\alpha_{\mu'} - \xi^\alpha_{,\mu'})(\delta^\beta_{\nu'} - \xi^\beta_{,\nu'})[g_{\alpha\beta}(x') - g_{\alpha\beta,\delta}(x)\xi^\delta]$$

$$= g_{\mu\nu}(x) - \xi^\alpha_{,\mu} g_{\alpha\nu} - \xi^\beta_{,\nu} g_{\mu\beta} - \underbrace{g_{\mu\nu,\delta}}_{=0}\xi^\delta$$

So we have freedom

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \xi_\mu}{\partial x^\nu} - \frac{\partial \xi_\nu}{\partial x^\mu}$$

Using it we set

$$\frac{\partial \bar{h}^\mu{}_\nu}{\partial x^\mu} = 0, \quad \text{where } \bar{h}^\mu{}_\nu = h^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu h \qquad (*)$$

with this condition

$$R_{\mu\nu} = \frac{1}{2}\Box h_{\mu\nu}, \quad \text{where } \Box = \Delta - \frac{1}{c^2}\frac{\partial}{\partial t^2}$$

Still, not all freedom removed. We have condition

$$\Box \xi^\mu = 0 \qquad (**)$$

Equation for grav. wave in vacuum:

$$\Box h^\mu{}_\nu = 0$$

solutions

$$h^\mu{}_\nu = Re(A^\mu{}_\nu e^{ik_\alpha x^\alpha}) = f(t \pm \frac{r}{c})$$

10 components = 4 diag + 6 off diag

condition $(*)$ removes 4

Condition $(**)$ or $x'^\mu = x^\mu + \xi^\mu(t - x/c)$

removes another 4.

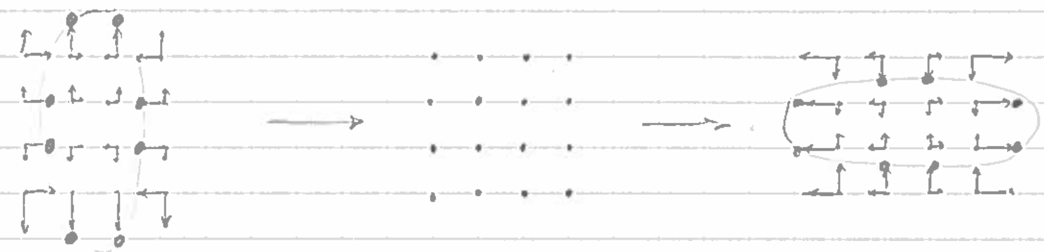
When done carefully:

$$h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2h_+ & 2h_\times & 0 \\ 0 & 2h_\times & -2h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So the metric associated with grav. wave propagation,

$$ds^2 = -dt^2 + [1 + Re(2h_+ e^{ik_\alpha x^\alpha})](dx^1)^2 + 4Re(h_\times e^{ik_\alpha x^\alpha})dx^1 dx^2 + [1 - Re(2h_+ e^{ik_\alpha x^\alpha})](dx^2)^2 + (dx^3)^2$$

+:

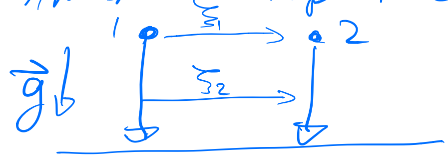


× Same but at 45° degres

Elias' favorite example

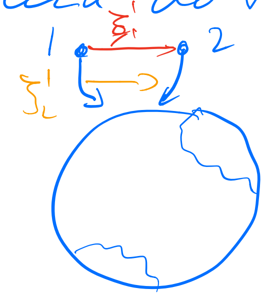
How to "measure" gravity?

A simple experiment: Drop two apples, at a distance $\vec{\xi}$ apart



Then for $\vec{g} = \text{const}$, $\vec{\xi}_1 = \vec{\xi}_2$. In other words, the apples fall in straight lines, while keeping the same distance.

Let's go into an orbit around the Earth and do the same:



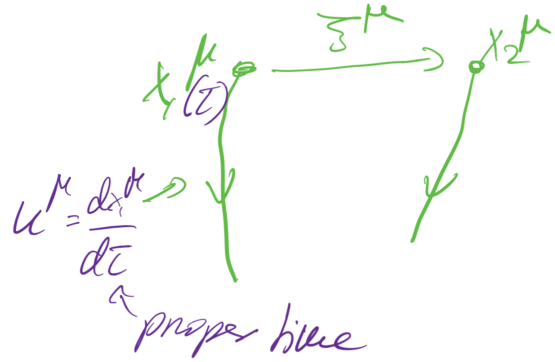
Now $\xi_1' \neq \xi_2'$ because of the appearance of tidal forces in the gravitational field.

Given the Newtonian potential Φ

$$\Rightarrow \quad \boxed{\ddot{\xi}_i = - \partial_i \partial_j \Phi \, \xi^j}$$

Measuring relative changes in the separation of the trajectories of two "freely falling" objects allows to probe the gravitational potential Φ!

We can now reformulate this in terms of the curvature of 4-dim manifolds.
Here the trajectories then represent the deviation of geodesics due to the 4-dim curvature of the spacetime



$$\boxed{u^\mu \nabla_\mu (u^\nu \nabla_\nu \xi^\lambda) = -R^\lambda{}_{\nu\mu\rho}\, \xi^\mu u^\nu u^\rho}$$

Equation of geodesic deviation

When performing such an experiment on Earth, we can safely assume to be in a locally flat frame

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}\left(\frac{|\xi|^2}{R}\right), \quad R \sim \frac{1}{(R_{\alpha\beta\mu\nu})}$$

unless the separation of these two test masses becomes too large.

This can be quantified by comparing the separation $|\xi|$ to the local radius of curvature $\mathcal{R}$.

In this frame, we have $u^\mu = (-1, 0, 0, 0)$

$u^\mu \nabla_\mu = -\partial_t$

$$\Rightarrow \quad \boxed{\ddot{\xi}^j = -R^j{}_{0i0}\,\xi^i}$$

Compare with the Newtonian expression!

This can now directly probe the Riemann tensor, i.e. the curvature of spacetime.

For a gravitational wave it can be shown that

$$R_{j0i0} = -\frac{1}{2}\ddot{h}_{ij}^{TT} \quad \text{(in transverse traceless gauge)}$$

$$\Rightarrow \quad \boxed{\ddot{\xi}^j = \frac{1}{2}\ddot{h}^j{}_i{}^{TT}\,\xi^i}$$

This allows us to directly observe perturbations caused by the gravitational wave!

Since these are small $\left(\ddot{h}_{ij}^{TT}\right) \ll 1$

$\sim 10^{-21}$! For compact binaries

at ~100 Mpc

$\Rightarrow \quad \ddot{\xi}^i \approx \frac{1}{2}\ddot{h}^{i\,TT}_{j}\, \xi^j(0)$

$\Rightarrow \quad \boxed{\frac{\Delta \xi}{\xi} \sim |h|}$

For a ground based GW detector

$\xi \simeq 4\,km \quad , \quad h \sim 10^{-21}$

Then $\underline{\Delta \xi \sim 10^{-16}\,cm}$

Measuring this is challenging!

Emission and energy of GW

It is clear that any motion of non-spherical mass distribution perturbs the metric and launches gravitational waves

Let I_{ij} be moment of inertia tensor
Quadruple momentum of the system

$$Q_{ij} = \int \left(y_i y_j - \frac{1}{3} y_k y_k \delta_{ij} \right) \rho \, \underbrace{d^3 y}_{\text{spatial}} = I_{ij} - \frac{1}{3} I_{kk} g_{ij}$$

In the far field

$$h_{00}^{GW} = 0$$
$$h_{0i}^{GW} = 0$$
$$h_{ij}^{GW} = \frac{1}{R} \left(2\ddot{Q}_{ij} + n_k n_e \ddot{Q}_{ke} \delta_{ij} + n_i n_j n_k n_e \ddot{Q}_{ke} - 2 n_j n_k \ddot{Q}_{ik} - 2 n_i n_k \ddot{Q}_{jk} \right)$$

Energy-momentum pseudo-tensor

$$t^{00} = t^{33} = -t^{03} = \frac{1}{4\pi} \left(2 h_+ \ddot{h}_+ + \dot{h}_+^2 + 2 h_\times \ddot{h}_\times + \dot{h}_\times^2 \right)$$
other comp. 0.

$$\langle t^{00} \rangle = \frac{1}{32\pi} \langle \underbrace{\dot{h}_{ij}^{GW} \dot{h}_{ij}^{GW}}_{\text{transverse - traceless}} \rangle$$

$$-\langle \dot{E} \rangle = \int_{\partial V} \langle t^{0i} \rangle n_i \, d^2 x = \frac{1}{5} \langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle$$

Angular momentum carried away

$$-\langle \dot{S}_i \rangle = \frac{2}{5} \epsilon_{ijk} \langle \ddot{Q}_{ej} \dddot{Q}_{ek} \rangle$$

III. APPLICATION: INSPIRAL OF A BINARY STAR

As a final application, let us consider the evolution of a binary star composed of two components with masses M_1 and M_2 with separation a on a circular orbit. We will make the velocities involved nonrelativisitic. The system has a kinetic+potential energy of

$$E_{\rm orb} = -\frac{M_1 M_2}{2a} \tag{33}$$

and hence a total mass of

$$M = M_1 + M_2 - \frac{M_1 M_2}{2a}. \tag{34}$$

The orbital frequency of the system is

$$\Omega \equiv \frac{2\pi}{P} = \frac{(M_1+M_2)^{1/2}}{a^{3/2}}. \tag{35}$$

Our interest is in following the effect of gravitational radiation on the orbit. To do this, we first need to find the quadrupole moment. For masses separated at angle $\phi = \phi_0 + \Omega t$, this is

$$Q_{ij} = \frac{M_1 M_2}{M_1+M_2} a^2 \begin{pmatrix} \cos^2\phi - \frac{1}{3} & \cos\phi\sin\phi & 0 \\ \cos\phi\sin\phi & \sin^2\phi - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}. \tag{36}$$

If we use the double-angle identities, this becomes

$$Q_{ij} = \frac{M_1 M_2}{M_1+M_2} a^2 \begin{pmatrix} \frac{1}{2}\cos 2\phi + \frac{1}{6} & \frac{1}{2}\sin 2\phi & 0 \\ \frac{1}{2}\sin 2\phi & -\frac{1}{2}\cos 2\phi - \frac{1}{6} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, \tag{37}$$

and taking the third derivative gives

$$\dddot{Q}_{ij} = \frac{M_1 M_2}{M_1+M_2} a^2\Omega^3 \begin{pmatrix} 4\sin 2\phi & 4\cos 2\phi & 0 \\ 4\cos 2\phi & -4\sin 2\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{38}$$

The gravitational wave power is then

$$-\langle \dot{E} \rangle = \frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle = \frac{1}{5}\left(\frac{M_1 M_2}{M_1+M_2}a^2\Omega^3\right)^2 [32] = \frac{32}{5}\left(\frac{M_1 M_2}{M_1+M_2}\right)^2 a^4\Omega^6. \tag{39}$$

Using Kepler's second law to eliminate Ω gives

$$-\langle \dot{E} \rangle = \frac{32}{5}\frac{M_1^2 M_2^2 (M_1+M_2)}{a^5}. \tag{40}$$

This is the rate at which the system loses orbital energy. Assuming that the masses of the objects don't change (e.g. that there is no transfer of energy from the internal structure of the bodies into the orbit), we may equate this with the rate of change of orbital energy,

$$\langle \dot{E} \rangle = \partial_t\left(M_1 + M_2 - \frac{M_1 M_2}{2a}\right) = \frac{M_1 M_2}{2a^2}\dot{a}, \tag{41}$$

and hence obtain

$$\dot{a} = -\frac{64}{5}\frac{M_1 M_2 (M_1+M_2)}{a^3}. \tag{42}$$

The $-$ sign indicates that the two bodies spiral together.

Since the rate of inspiral due to gravitational wave emission is proportional to a^{-3}, it follows that as the two bodies approach each other, they inspiral faster and faster. One may find the approach time by taking

$$\partial_t(a^4) = 4a^3\dot{a} = -\frac{256}{5}M_1 M_2 (M_1+M_2), \tag{43}$$

and hence we see that the inspiral reaches $a=0$ in a finite time

$$t_{\rm GW} = \frac{5a^4}{256 M_1 M_2 (M_1+M_2)}. \tag{44}$$

This time is shortest for massive bodies on close orbits, as one might expect.

A. Examples

As a simple example, let's consider the inspiral times associated with solar-system scales. Recall that, converted into times, a solar mass is 4.9 μs and the astronomical unit is 500 s. Therefore, we can calculate the inspiral time of a system:

$$t_{\rm GW} = 3.3 \times 10^{17}\,{\rm yr}\,\frac{(a/1\,{\rm AU})^4}{M_1 M_2 (M_1 + M_2)/M_\odot^3}. \tag{45}$$

For the Earth orbiting the Sun, with $M_1 = M_\odot$ and $M_2 = 3 \times 10^{-6} M_\odot$ at a separation of 1 AU, the inspiral time is 10^{23} years. Of course by then the Sun will have turned into a white dwarf, Mercury and (maybe) Venus and Earth will have been consumed, and it is doubtful even that the orbits of the other planets are stable over that timescale. As a more extreme example one could consider the "hot Jupiters" that have been found around other stars with $M_1 \sim 10^{-3} M_\odot$ and $a = 0.05$ AU. There the inspiral time is 2×10^{15} years. So we can see that even in extreme situations, gravitational waves have no effect on planetary orbits.

Gravitational waves do however have a more significant effect on binary stars. If we consider a binary with masses of $M_1 = M_2 = M_\odot$, and we ask how close the orbits must be to merge in less than the age of the Universe (10^{10} years), we find

$$a < 0.016\,{\rm AU} \quad {\rm or} \quad P < 12\,{\rm hr}. \tag{46}$$

There are many instances of stellar remnants (white dwarfs and neutron stars) in orbits with periods of this order of magnitude or shorter (even as short as a few minutes). Such objects will spiral in due to gravitational wave emission and lead to mergers, which will be detectable as bursts of gravitational waves by the next generation of detectors.