

## HOW ACCURATELY CAN WE CALCULATE THE DEPTH OF THE SOLAR CONVECTIVE ZONE?

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### ABSTRACT

We evaluate the logarithmic derivative of the depth of the solar convective zone (CZ) with respect to the logarithm of the radiative opacity,  $\partial \ln R_{CZ} / \partial \ln \kappa$ . We use this expression to show that the radiative opacity near the base of the solar CZ must be known to an accuracy of  $\pm 1\%$  in order to calculate the CZ depth to the accuracy of the helioseismological measurement,  $R_{CZ} = 0.713 \pm 0.001 R_{\odot}$ . The radiative opacity near the base of the CZ that is obtained from OPAL tables must be increased by  $\sim 21\%$  in the 2004 Bahcall-Pinsonneault solar model if one wants to invoke opacity errors in order to reconcile recent solar heavy abundance determinations with the helioseismological measurement of  $R_{CZ}$ . We show that the radiative opacity near the base of the CZ depends sensitively on the assumed heavy-element mass fraction,  $Z$ . The uncertainty in the measured value of  $Z$  is currently the limiting factor in our ability to calculate the depth of the CZ. Different state-of-the-art interpolation schemes using the existing OPAL tables yield opacity values that differ by  $\sim 4\%$ . We describe the finer grid spacings that are necessary to interpolate the radiative opacity to  $\pm 1\%$ . Uncertainties due to the equation of state do not significantly affect the calculated depth of the CZ.

*Subject headings:* atomic data — convection — methods: numerical — Sun: interior

### 1. INTRODUCTION

The depth of the solar convective zone (CZ) has been measured by helioseismological techniques to high accuracy. In the most comprehensive study to date, Basu & Antia (1997) have investigated the influence of observational and theoretical systematic uncertainties, as well as measurement errors. Basu & Antia (1997) concluded that the base of the solar CZ currently lies at a depth of

$$R_{CZ} = 0.713 \pm 0.001 R_{\odot}. \quad (1)$$

The result of Basu & Antia (1997) is consistent with the earlier measurements of Kosovichev & Fedorova (1991), who obtained  $R_{CZ} = 0.713 \pm 0.001 R_{\odot}$ , and Christensen-Dalsgaard et al. (1991), who also obtained  $R_{CZ} = 0.713 \pm 0.003 R_{\odot}$ , as well as with the determination of Guzik & Cox (1993), who found  $R_{CZ} = 0.712 \pm 0.001 R_{\odot}$ . Basu (1998) also studied the effect of the assumed value of the solar radius on the inferred depth of the CZ and found  $R_{CZ} = 0.7135 \pm 0.0005 R_{\odot}$ . The analyses in these different studies span a wide range of reference solar models and analysis techniques.

On the basis of the analyses cited above, the measurement of the depth of the solar CZ appears robust and precise.

Recently, new precision measurements have been made of the C, N, O, Ne, and Ar abundances on the surface of the Sun (Allende Prieto et al. 2001, 2002; Asplund et al. 2000, 2004; Asplund 2000; M. Asplund 2003, private communication). These new abundance determinations use three-dimensional rather than one-dimensional atmospheric models, include hydrodynamical effects, and pay particular attention to uncertainties in atomic data and the observational spectra. The new abundance estimates, together with the previous best estimates for other solar surface abundances (Grevesse & Sauval 1998),

imply  $Z/X = 0.0176$ , much less than the previous value of  $Z/X = 0.0229$  (Grevesse & Sauval 1998).

For a solar model with the recently determined heavy-element-to-hydrogen ratio, the calculated depth of the CZ is (Bahcall & Pinsonneault 2004)

$$R_{CZ}(Z/X = 0.0176) = 0.726 R_{\odot}, \quad (2)$$

which is very different from the measured depth of the CZ (see eq. [1]). On the other hand, Basu & Antia (2004) have shown that the helioseismological determination of  $R_{CZ}$ , equation (1), is not affected if one assumes the correctness of the lower heavy-element abundances ( $Z/X = 0.0176$ ).

Something is wrong. We have a new solar problem: “the CZ problem.”

The radiative opacity is a key ingredient in calculating the depth of the CZ. Moreover, about 95% of the total radiative opacity near the base of the CZ involves bound electrons, either bound-free or bound-bound opacity (C. A. Iglesias 2004, private communication). Thus, opacity calculations in this region involve details of the ionization balance and other delicate atomic physics properties.

In this paper we focus on determining the accuracy with which the opacity near the base of the CZ must be known in order to calculate precisely the depth of the CZ with a stellar evolution code. We also evaluate the accuracy with which the opacity near the base of the CZ can be interpolated from OPAL tables. For a related comparison of the Los Alamos LEDCOP opacities and the OPAL opacities, see Neuforge-Verheucke et al. (2001). For comprehensive discussions of stellar radiative opacities, the reader is referred to the important reviews by Rogers & Iglesias (1998) and Seaton (1995).

We investigate in Bahcall et al. (2004) the helioseismological implications of the changes in opacity that are discussed in the

present paper. The viability of any proposed change in the opacity discussed in the present paper must be tested by comparing a solar model that is evolved with the changed opacity with a complete set of precise helioseismological data. There is no compelling reason to believe that the illustrative change in opacity considered here, which is highly peaked in radius, will be either reproduced exactly by new opacity calculations or precisely consistent with helioseismological constraints. In the future, once new opacity calculations are available that satisfy the requirements described in this paper, it will be possible to test simultaneously the new opacities, the solar model evolution, and the helioseismological implications.

We derive in § 2 the dependence,  $\partial \ln R_{CZ} / \partial \ln \kappa$ , of the calculated depth of the solar CZ on the assumed radiative opacity. We apply this result to determine the accuracy with which the opacity must be known in order to calculate the depth of the CZ to the accuracy with which it is measured helioseismologically. We also determine the change in the standard OPAL opacity that is required in order to reconcile the helioseismological measurement with the recent determinations of heavy-element abundances. We evaluate in § 3 the dependence of the radiative opacity near the base of the CZ on the stellar composition. We find that the opacity depends sensitively on the assumed heavy-element abundance. We compare in § 4 the opacities obtained from two different interpolation schemes that are both applied to the same published OPAL opacity tables. Throughout this paper we adopt the OPAL opacities (Iglesias & Rogers 1991a, 1991b, 1996; Rogers & Iglesias 1992) as standard, when supplemented at low temperatures by values from Alexander & Ferguson (1994). We use simulated opacity tables in § 5 to estimate the likely uncertainties that result from interpolations within the existing OPAL opacity tables and to determine the grid sizes to obtain small interpolation errors. For completeness and for contrast, we use four different equations of state (EOSs) to show in Appendix A that uncertainties due to the choice of EOS are not important, at the present level of accuracy, for the calculation of the depth of the solar CZ. We also demonstrate in Appendix B that uncertainties in the nuclear reaction rates affect the depth of the solar CZ only at the level of 0.1%. In Appendix C we verify that the conversion of carbon and oxygen in CNO burning, which cannot be accurately modeled with existing opacity codes, causes a 0.1% uncertainty in the calculated depth of the CZ. Basu & Antia (2004; see also Asplund et al. 2004) have shown that errors in the calculation of the diffusion coefficients are unlikely to be the correct explanation of the discrepancy between measured and calculated depth of the solar CZ. Other solar model ingredients, including the element diffusion coefficients, can affect the calculated depth of the CZ. A complete investigation of all the possible effects on the CZ is beyond the scope of the present paper and would distract the reader from our main concern, the effect of the radiative opacity. Moreover, we believe that the radiative opacity and the heavy-element abundance provide the single largest contributions to the error budget for the calculation of the solar CZ. The effect of the heavy-element abundance on the calculated depth of the CZ has been evaluated in Bahcall & Pinsonneault (2004). We summarize and discuss our main results in § 6.

## 2. DEPENDENCE OF CALCULATED DEPTH OF CONVECTIVE ZONE ON RADIATIVE OPACITY

In this section we determine the dependence of the calculated depth of the solar CZ on the value of the radiative opacity in the vicinity of the base of the CZ. Using this

TABLE 1  
ADOPTED COMPOSITIONS

Element	BP04+	BP04	Element	BP04+	BP04
C.....	8.39	8.52	Cl.....	5.28	5.28
N.....	7.80	7.92	Ar.....	6.18	6.40
O.....	8.69	8.83	K.....	5.13	5.13
Ne.....	7.84	8.08	Ca.....	6.35	6.35
Na.....	6.32	6.32	Ti.....	4.94	4.94
Mg.....	7.58	7.58	Cr.....	5.69	5.69
Al.....	6.49	6.49	Mg.....	5.53	5.53
Si.....	7.56	7.56	Fe.....	7.50	7.50
P.....	5.56	5.56	Ni.....	6.25	6.25
S.....	7.20	7.20			

NOTES.—The adopted compositions used for the computation of solar models BP04+ and BP04 (and variations thereupon). The relative abundances given in the table denote  $\log N_i$  in the usual scale in which  $\log N_H = 12$ . We use meteoritic abundances when available. In the past, when conflicts between meteoritic and atmospheric abundances have existed, the meteoritic determinations have often turned out to be more correct.

dependence, we then answer two questions. First, how accurately must the radiative opacity be known in order to calculate the depth of the CZ to the accuracy with which the depth is measured by helioseismology? Second, how large must the error in the radiative opacity at the base of the solar CZ be in order to explain the difference between the measured value of  $R_{CZ}$  and the value of  $R_{CZ}$  that is calculated in a solar model that is constructed using the recently determined heavy-element abundances ( $Z/X = 0.0176$ )?

We follow the approach introduced by Bahcall et al. (1969) in which we change the standard (OPAL) opacity in the vicinity of the CZ by a small functional amount that depends on an adjustable parameter  $\alpha$ . We calculate a series of solar models for different values of  $\alpha$ , which permits us to evaluate the logarithmic derivative of  $R_{CZ}$  with respect to the opacity near the base of the CZ. We begin with a brief description of the solar models used in our studies.

### 2.1. Description of the Solar Models

The solar age adopted in this article is  $4.57 \times 10^9$  yr. At this age, the solar models are required to have the present values for the solar luminosity ( $L_\odot$ ), the solar radius ( $R_\odot$ ), and the ratio of heavy elements to hydrogen by mass ( $Z/X$ ). We adopt the values  $L_\odot = 3.8418 \times 10^{33}$  ergs $^{-1}$ ,  $R_\odot = 6.9598 \times 10^{10}$  cm, and  $Z/X = 0.0229$ , respectively (see Bahcall et al. 2001). The models are calculated using the OPAL EOS (hereafter OPAL 1996; Rogers et al. 1996), unless stated otherwise, and the OPAL opacities (see § 1). The nuclear reaction rates are those used in Bahcall et al. (2001). Element diffusion is incorporated for helium and metals (Thoul et al. 1994). We use the mixing-length theory for convection and the Schwarzschild criterion to determine the location of the convective boundaries.

The adopted heavy-element composition is, as discussed in Bahcall & Pinsonneault (2004), one of the most important ingredients in determining the value of the solar CZ that is obtained from a stellar evolution code. For specificity, we show in Table 1 the specific composition that has been adopted in computing the models referred to as solar model BP04+ (see Bahcall & Pinsonneault 2004; includes recent composition determinations described in Allende Prieto et al. 2001, 2002; Asplund et al. 2000, 2004; Asplund 2000) and solar model BP04 (see Bahcall & Pinsonneault 2001; composition from

Grevesse & Sauval 1998). OPAL opacities have been computed for the same compositions. The atomic masses on the OPAL Web site were used in conjunction with these abundances to compute  $Z/X$ . Although the most precise details of the composition are not important for the general issues discussed in this paper, Table 1 permits us to make clear exactly what compositions were used in the calculations described in this paper. This may be helpful to the reader since new composition determinations are currently appearing at frequent intervals.

For the BP04+ solar model, the initial (final) mass fractions are  $X_0 = 0.71564$  (0.74862),  $Y_0 = 0.26960$  (0.23817), and  $Z_0 = 0.01476$  (0.01321). For the BP04 model, the corresponding mass fractions are  $X_0 = 0.70775$  (0.72465),  $Y_0 = 0.27344$  (0.24335), and  $Z_0 = 0.01881$  (0.01692). Note that the helium abundances in the two models are the same to within about 2%, although the heavy-element mass fractions differ by more than 25%.

One of the main goals of this paper is to compare the numerical results obtained for different solar evolution codes. To this end, we compare the results obtained with the Bahcall-Pinsonneault/Yale code (see Bahcall & Pinsonneault 2004 and references therein) with the Garching/Weiss stellar evolution code. For further details about the Garching stellar evolution code, we refer the reader to Schlattl et al. (1997), Schlattl (2002), and references therein.

## 2.2. Evaluation of $\partial \ln R_{CZ} / \partial \ln \kappa$

For relatively small changes in the radiative opacity, the sensitivity to opacity of the calculated depth of the solar CZ can be expressed in terms of a single numerical parameter  $\beta$ , which is defined by the relation

$$\beta \equiv \frac{\partial \ln R_{CZ}}{\partial \ln \kappa}. \quad (3)$$

To evaluate  $\beta$ , we multiply the OPAL opacity in the vicinity of the convective envelope boundary by a Lorentzian function  $f(T)$  given by

$$f(T) = 1 + \frac{\alpha \gamma^2}{(T - T_0)^2 + \gamma^2}. \quad (4)$$

Here  $T$  is the temperature in the solar model. We label each radial point in the solar model by its corresponding value of  $T$ . We can then write for the opacity that  $\kappa = \kappa_0 f(T)$ , where  $\kappa_0$  is the unperturbed radiative opacity,  $\alpha$  is the amplitude of the perturbation, and  $\gamma$  is the width of the perturbation (defined as the point where the perturbation drops to  $\alpha/2$ ).

At the present solar age, the temperature at the base of the CZ is  $T \approx 2.18 \times 10^6$  K, so this value is adopted for  $T_0$ . We calculated solar models for  $\gamma = 0.2 \times 10^6$  K  $\approx 0.1 T_0$  and  $\alpha = 0, \pm 0.030, \pm 0.060$ , which were well represented by a fixed value of  $\beta$ .

We find

$$\beta = -0.095 = \frac{\partial \ln R_{CZ}}{\partial \ln \kappa}, \quad (5)$$

or, equivalently,

$$\left( \frac{R_{CZ}}{R_{CZ,0}} \right) = \left( \frac{\kappa}{\kappa_0} \right)^{-0.095}. \quad (6)$$

Since we have used converged solar models that satisfy the observational constraints on the luminosity, the chemical composition, and other parameters, the result given in equation (6) includes all of the feedback effects required by the boundary conditions and the external observational constraints.

To test the robustness of this result, we doubled the value of  $\gamma$  to  $\gamma = 0.4 \times 10^6$  K and obtained for this broader perturbation a similar value for  $\beta$ , namely,  $\beta = -0.10$  (instead of  $-0.095$ ). Of course, we do not know a priori the exact form of any future change in the radiative opacity that may result from further research. Nevertheless, we can conclude from the examples we have studied that equation (6) is a good approximation to changes in the opacity that are local and peaked at the base of the CZ.

The sign of  $\beta$ , which is given in equation (5), is evident from physical reasoning. The magnitude of the radiative temperature gradient is proportional to the opacity since the radiative flux passing through a given point in the star is fixed. If the radiative opacity is increased at a fixed point, then the radiative gradient is increased and the condition for convective stability becomes more difficult to satisfy. The star can become convectively unstable at a smaller radius (higher temperature).

The changes in opacity considered here will necessarily bring about small changes in the inferred surface mass fraction of hydrogen. Quantitatively, we find analogous to equation (5) that  $(X/X_0) = (\kappa/\kappa_0)^{-0.023}$ , i.e., a weak dependence. A change of 20% in opacity leads to an estimated change in  $X$  of about 0.4%, less than the uncertainties in the helioseismological determinations of  $X$ .

## 2.3. How Accurately Do We Need to Know the Opacity?

Equations (5) and (6) imply that in order to calculate the depth of the CZ to the accuracy with which the depth is measured, 1 part in 713, one must know the radiative opacity at the base of the CZ to an accuracy of

$$\left( \frac{\Delta \kappa}{\kappa} \right)_{\text{equivalent experimental accuracy}} = 1\%. \quad (7)$$

This is extremely high precision for a calculated radiative opacity, probably beyond the reach of existing calculations.

If we try to explain the difference between the measured value of  $R_{CZ}$  (see eq. [1]) and the value calculated using recently determined heavy-element abundances (see eq. [2]), then we find that the opacities used in the solar model must be in error by

$$\left( \frac{\Delta \kappa}{\kappa} \right)_{Z/X=0.0176} = 21\%. \quad (8)$$

The result shown in equation (8) applies for the Bahcall & Pinsonneault (2004) solar model BP04. The Garching code leads to a slightly larger discrepancy between calculated and measured depth of the CZ (see § 4).

We have evolved a solar model based on the recent abundance determinations, BP04+, but with a 21% increased opacity near the base of the CZ. The calculated depth of the CZ is  $R_{CZ} = 0.7133 R_{\odot}$ , in good agreement (by design) with the measured value. The initial (final, surface) mass fractions for this model are  $X_0 = 0.71621$  (0.74776),  $Y_0 = 0.26919$  (0.23908), and  $Z_0 = 0.01460$  (0.01316). The current surface abundance of  $Y$  implied by this model is about  $3 \sigma$  smaller than the value determined by Basu & Antia (2004) from

helioseismology. We are not sure how to regard this discrepancy since the overwhelmingly dominant error in the helioseismological value is systematic, not statistical. In the forthcoming paper by Bahcall et al. (2004), we will compare the BP04+ solar model with increased opacity with all of the available helioseismological data.

The estimate given in equation (8) is based on the assumption that the opacity is changed only locally, i.e., near the base of the CZ (see eq. [4]). If, instead, one changes the opacity by changing the surface heavy-element abundance,  $Z/X$ , then the opacity is affected throughout the solar model and the change required near the base of the CZ can be different. The inputs to the models BP04 and BP04+ of Bahcall & Pinsonneault (2004) differ only in the assumed heavy-element abundance. BP04+ was calculated assuming  $Z/X = 0.0176$  (recently determined low heavy-element abundance), and BP04 was calculated using  $Z/X = 0.0229$  (Grevesse & Sauval 1998). Using the results obtained from these two models (comparing the calculated difference in the CZ depth between the two converged solar models with the difference in radiative opacity at the same  $T$  and  $\rho$  near the base of the CZ), we estimate that the opacity near the base of the CZ must change by  $\simeq 14\%$  if the pattern of opacity changes is similar to that induced by composition changes.

We evaluate in the next section the sensitive dependence of the radiative opacity on the assumed surface heavy-element abundance.

### 3. DEPENDENCE OF RADIATIVE OPACITY ON COMPOSITION

In this section we estimate the dependence of the radiative opacity near the base of the CZ on the stellar composition. We approximate the opacity as a function of the hydrogen mass fraction,  $X$ , and the heavy-element mass fraction,  $Z$ . Thus,  $\kappa = \kappa(X, Z)$ .

The fractional uncertainty in the opacity may then be written in the form

$$\frac{d\kappa}{\kappa} = \left( \frac{\partial \ln \kappa}{\partial \ln Z} \right)_X \frac{dZ}{Z} + \left( \frac{\partial \ln \kappa}{\partial \ln X} \right)_Z \frac{dX}{X}. \quad (9)$$

We have used the existing OPAL opacity tables to evaluate numerically the fractional derivatives that appear in equation (9). We find

$$\left( \frac{\partial \ln \kappa}{\partial \ln Z} \right)_X \simeq 0.70, \quad \left( \frac{\partial \ln \kappa}{\partial \ln X} \right)_Z \simeq 0.15. \quad (10)$$

The numerical values for the logarithmic derivatives given in equation (10) were determined for conditions similar to those at the base of the current solar CZ; we used  $\log T = 6.34$ ,  $\log \rho = -0.7$ ,  $X = 0.74$ , and  $Z = 0.0169$ . Changing the values of the physical variables at which the derivatives are evaluated causes only small changes in the derivatives as long as the changes are restricted to stellar positions close to the base of the CZ.

The uncertainty in the opacity is dominated by the uncertainty in the heavy-element abundance,  $Z$ . If we want to know the opacity to 1%, the accuracy required to calculate the depth of the solar CZ to the precision with which the depth is measured (see eq. [7]), then we have to determine the surface heavy-element abundance to a precision of 1%. This seems like, at present, an impossibly ambitious demand, at least for

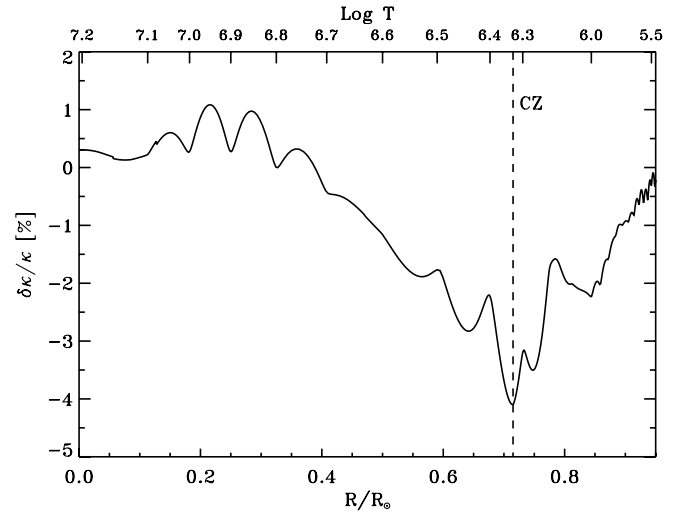


FIG. 1.—Fractional difference in the opacity,  $\delta\kappa/\kappa$  (in percent), obtained from two interpolation schemes embodied in two widely used stellar evolution programs, the Yale/BP code and the MPA code. The fractional difference is defined to be  $\delta\kappa/\kappa = (\text{result from birational spline} - \text{result from four-point Lagrangian}) / \text{result from four-point Lagrangian}$ . The figure was made for a fixed  $T$ ,  $\rho$ ,  $X$ ,  $Z$  profile so the differences that are shown are only due to interpolation.

the foreseeable future. The current  $1\sigma$  uncertainty in  $Z$  is about 15% (see Bahcall & Pinsonneault 2004).

In the next two sections we estimate how accurately the radiative opacity can be interpolated from the existing OPAL opacity tables.

### 4. COMPARISON OF THE RADIATIVE OPACITY OBTAINED FROM TWO DIFFERENT INTERPOLATION SCHEMES

We compare in this section the radiative opacity values interpolated from standard OPAL opacity tables by two different interpolation schemes embedded in two extensively used state-of-the-art stellar evolution codes. In particular, we interpolate within the OPAL tables using a four-point Lagrangian scheme that is implemented in the Yale/BP stellar evolution code (Guenther et al. 1989; Pinsonneault et al. 1989; Bahcall & Pinsonneault 1992, 1995, 2001) and a birational spline scheme (Späth 1995) that is implemented in the Garching code (Schlattl & Weiss 1997).

The implementations of these two different interpolation schemes have been extensively tested. There is no absolute way to evaluate their accuracy since the precision that is obtained depends on the behavior of the function being interpolated. The two interpolation schemes have different advantages and disadvantages.

Figure 1 shows the fractional difference,  $\delta\kappa/\kappa$ , between the radiative opacity that is obtained using the Yale/BP four-point Lagrangian scheme and the opacity interpolated using the Garching birational spline scheme (the damping parameter for the birational splines was set to 5). The fractional opacity is displayed at different radii (*lower horizontal scale*) and at different temperatures (*upper horizontal scale*) in a standard solar model. The figure was made for a fixed  $T$ ,  $\rho$ ,  $X$ ,  $Z$  profile so the differences that are shown are only due to interpolation.

The amplitude of the difference becomes as large as 4% near the base of the CZ, which is denoted by a vertical line in Figure 1. The interpolated value of the opacity near the base of the CZ is particularly sensitive to the interpolation scheme because the temperature of the solar CZ and the value of

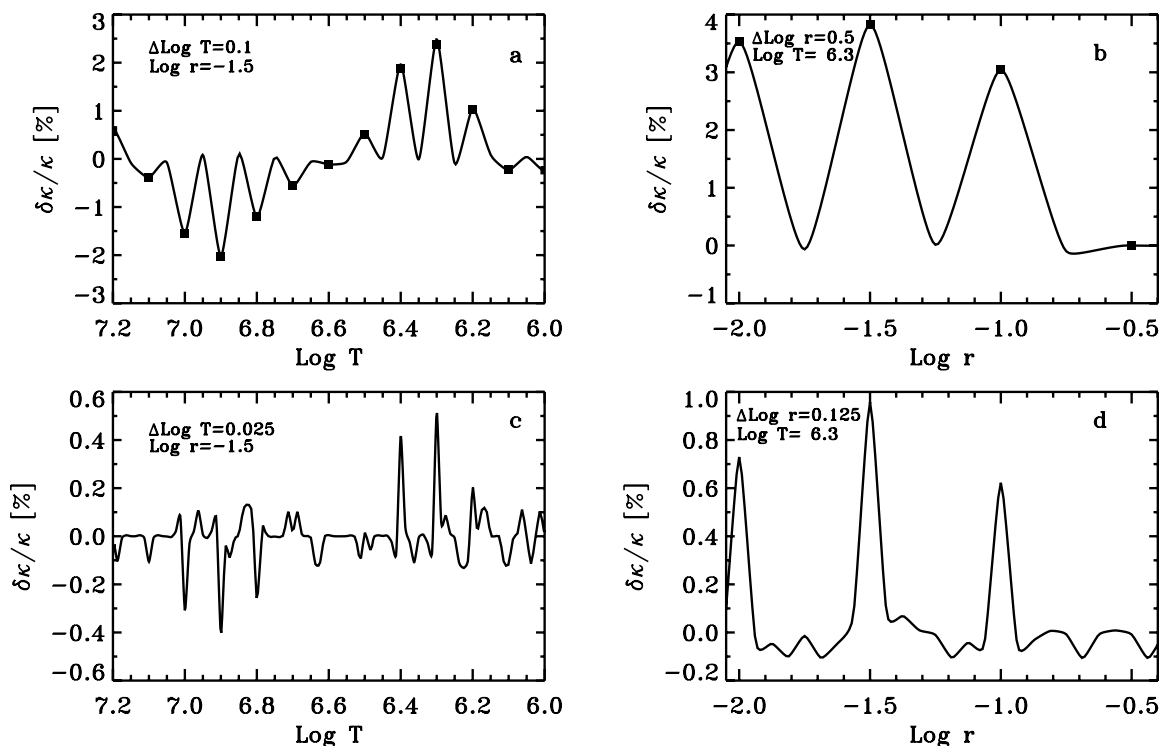


FIG. 2.—Relative opacity differences,  $\delta\kappa/\kappa = (\kappa_{\text{std}} - \kappa_{\text{shifted}})/\kappa_{\text{standard}}$ , that were found between the interpolated opacities that were obtained using the *shifted* and the *standard* opacity tables (see text for explanation) as a function of the interpolation variable. Panel *a* uses the original OPAL grid spacing and compares the results with the values obtained from shifting the tables in temperature ( $\Delta \log T$ ). Panel *b* is analogous to *a*, but the opacity tables are shifted in  $r$  (defined as  $r = \rho/T_6^3$ , shifted in  $\Delta \log r$ ). The squares denote the location of the tabulated values in the original OPAL opacity tables. Panels *c* and *d* are similar to *a* and *b*, respectively, but with grid spacings reduced by a factor of 4. Although the regions for  $\log T < 6.3$  ( $\log r > -1.5$ ) are inside the convective envelope, they are shown for the sake of clarity. In each case, the grid spacing is given, together with the fixed value for the other variable. In all cases the hydrogen and metal mass fractions are fixed at  $X = 0.7$  and  $Z = 0.02$ , respectively.

$r = \rho/T_6^3$  fall about halfway between two points at which the OPAL opacity is tabulated.

The amplitude of the discrepancy between the two interpolation schemes is much larger than is permitted if one wants to calculate the depth of the CZ to the measured accuracy (see eq. [7]). The above discussion shows that uncertainties due to interpolation contribute importantly to the error budget for the calculation of the solar CZ (see eq. [8]).

### 5. SIMULATED OPACITY TABLES: COMPARISON OF INTERPOLATED VALUES WITH STANDARD VALUES

In this section we make plausible estimates of the uncertainties inherent in interpolating the radiative opacity from the available OPAL opacity tables. We use simulated opacity tables to make self-consistency tests of the accuracy of the interpolation schemes we use. For specificity, we cite the results obtained using the birational spline. Similar results were found with the four-point Lagrangian spline. The inferences obtained in this section regarding the accuracy of interpolation within opacity tables complement and supplement the results obtained in § 4 by comparing the outputs of two different interpolation schemes.

The figures that we show are based on the following strategy. Using the existing OPAL opacity tables, we interpolate the value of the opacity at shifted points, making in this way new but simulated tables. We then use the simulated tables to predict the values of the opacity at the original, unshifted points. We take as one measure of the likely uncertainty the difference between the opacity values in the original published tables and the opacity values at the same

points obtained by interpolating in the simulated tables. In all cases, the largest errors are expected (and observed) at points originally tabulated in OPAL tables because none of the shifted tables (independent of the grid spacing) have these points tabulated. Since the originally tabulated values lay in the middle of the shifted intervals, they give the largest errors.

We have also tested the accuracy of the interpolation schemes by artificially making the OPAL tables more sparse, i.e., by omitting points. We then interpolate in the sparser tables to see how well the interpolation reproduces the omitted values. The uncertainty estimates obtained using sparser tables are in good agreement with the uncertainties obtained by shifting points. We concentrate our discussion here on the results found with the shifted tables because these results are more easily displayed.

Figure 2 shows the fractional differences in the opacity,  $\delta\kappa/\kappa$ , that were found between the values given in the original OPAL tables and the values that were obtained from the simulated shifted tables. Figures 2*a* and 2*b* use the actual grid spacing of the OPAL tables. The OPAL tables are presented in terms of the logarithm of the temperature ( $\log T$ ) and the logarithm of  $r \equiv \rho/T_6^3$  ( $\log r$ ).

Figures 2*a* and Figure 2*b* show that the amplitudes of  $\delta\kappa/\kappa$  can be as large as 3% in interpolating within the shifted OPAL tables.

How dense does the opacity grid have to be in order for the interpolation uncertainty within the grid to be less than 1% (see eq. [7]) for opacities near the base of the CZ? In order to provide a plausible answer to this question, we have

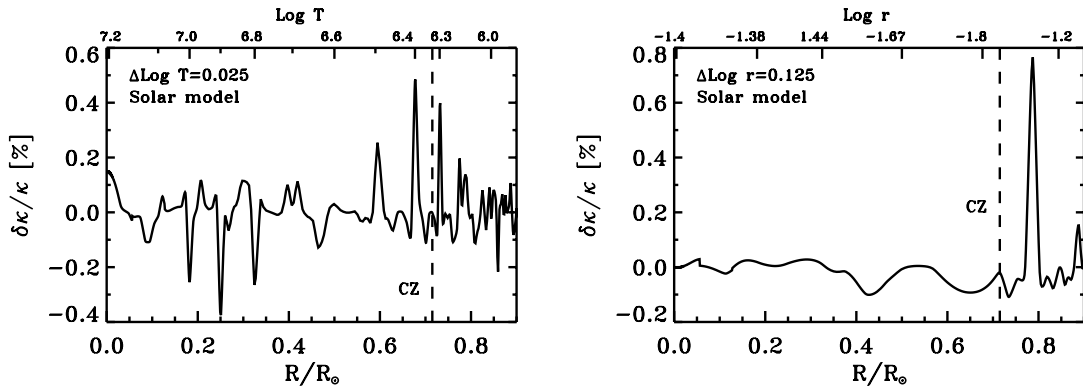


FIG. 3.—Estimated fractional uncertainties,  $\delta\kappa/\kappa$ , in interpolating for a solar model the radiative opacities in simulated opacity tables with a grid size  $\Delta \log T = 0.025$ ,  $\Delta \log r = 0.125$ , and with heavy-element composition values ranging from  $Z = 0.0100$  to  $0.0225$  with  $\Delta Z = 0.0025$ . The left panel has  $\Delta \log T = 0.025$  and  $\Delta \log r = 0.5$  (original spacing), and the right panel has  $\Delta \log T = 0.1$  (original spacing) and  $\Delta \log r = 0.125$ . The opacities obtained by interpolating in shifted simulated tables in  $\Delta \log T$ ,  $\Delta \log r$ , and  $Z$  are compared with the values obtained in unshifted simulated tables. The differences  $\delta\kappa/\kappa$  are shown as a function of the fractional solar radius,  $R/R_\odot$ . The upper horizontal axis shows the corresponding values of  $\log T$  ( $\log r$ ) for the left (right) panel. The location of the base of the CZ zone is shown by a vertical dotted line.

interpolated within simulated opacity tables with grids of a variety of different densities.

Figures 2c and 2d show the results of interpolations within simulated tables that have grid sizes, respectively, of  $\Delta \log T = 0.025$  and  $\Delta r = 0.125$ . These simulated tables are 4 times as dense in each variable as the existing OPAL opacity tables. The errors in predicting the unshifted opacity values in the original OPAL tables are less than 0.6% (throughout the physically relevant region) when the values in the simulated shifted tables are used. With the simple assumptions we have made, the estimated errors scale approximately linearly with the grid spacing. However, this linear dependence results in large part from our assumption that the opacity values are smooth in  $\log T$  and  $\log r$  in the regions of interest.

The situation is somewhat different for the heavy-element abundance,  $Z$ . The existing OPAL opacity tables present values for three heavy-element abundances relevant to the Sun:  $Z = 0.01, 0.02, 0.03$ . However, recent redeterminations of the heavy-element abundance in the Sun have suggested that  $Z$  is significantly lower than was previously believed (Allende Prieto et al. 2001, 2002; Asplund et al. 2000, 2004; Asplund 2000). We have experimented numerically with interpolating in  $Z$  within the existing OPAL opacity tables and also within simulated opacity tables with a denser grid in  $Z$ . We find that the required accuracy (better than 1%) in interpolation can be achieved if a grid with  $\Delta Z = 0.0025$  is used for values of  $Z$  ranging from  $Z = 0.0100$  to  $0.0225$ . This amounts, in addition to a denser grid, to a shift to lower values in the mean value of  $Z$  that is tabulated. Fortunately, we find that the existing OPAL grid in the hydrogen abundance,  $X$ , is sufficient to permit interpolation in the opacity to the required accuracy.

Figure 3 displays for the points in a standard solar model the expected uncertainties in interpolating the radiative opacity within opacity tables that have our preferred grid spacings, namely,  $\Delta \log T = 0.025$  and  $\Delta r = 0.125$ . In computing the expected uncertainties shown in Figure 3, we created simulated OPAL tables at shifted points in all three variables:  $T$ ,  $r$ , and  $Z$ . We then used the simulated tables to calculate the opacity at points (defined by  $T$ ,  $r$ , and  $Z$ ) that correspond to points in the standard solar model. The opacity obtained from shifted simulated tables was compared with the opacity obtained from unshifted simulated opacity tables. The differences,  $\Delta\kappa/\kappa$ ,

between shifted and unshifted simulated opacities are plotted in Figure 3 as a function of the radial position,  $R/R_\odot$ , in the solar model and also as a function of the corresponding values of either  $\log T$  or  $\log r$ . The dotted vertical line indicates the location of the base of the CZ.

We conclude from Figures 2 and 3 that opacity tables with grid sizes of  $\Delta \log T = 0.025$  and  $\Delta r = 0.125$  are probably accurate enough to permit a precise calculation of the depth of the solar CZ using existing stellar evolution codes. For the dense grid sizes considered here, the interpolations within the opacity tables should not cause errors that prevent an accurate calculation of the depth of the solar CZ. However, the absolute value of the tabulated radiative opacities could still introduce significant uncertainties.

## 6. SUMMARY AND DISCUSSION

The primary goal of this paper is to determine how accurately the radiative opacity near the base of the CZ must be known in order to use measurements of the CZ depth to draw conclusions about other solar parameters. There are two separate but related issues with respect to the accuracy of the radiative opacity, namely, the accuracy with which the tabulated values in opacity tables are calculated and the accuracy with which the opacity can be interpolated within tables of a specified grid size. We first summarize our conclusions regarding the accuracy of tabulated opacity values and then we summarize our results with respect to the accuracy of interpolations within the standard OPAL opacity tables. The helioseismological implications of the opacity changes considered in this paper will be discussed in Bahcall et al. (2004).

We show in § 2 that the logarithmic derivative of the CZ depth with respect to the logarithm of the opacity satisfies  $\partial \ln R_{CZ} / \partial \ln \kappa \approx -0.095$ . We conclude from this relation that the radiative opacity must be known to an accuracy of 1% in order to calculate in a solar model the depth of the CZ to the accuracy, 0.14%, with which the depth is measured by helioseismology. On the other hand, if one accepts the recent measurements of heavy-element abundances, then the OPAL opacities must be increased by about 21% in order to reconcile the calculated solar model depth of the CZ and the measured depth of the CZ. This change of 21% could conceivably arise from a combination of errors in the tabulated values of the opacity and interpolation errors, which are discussed below.

However, as we shall see, the total change of 21% is too large to be ascribed solely to errors in interpolation.

It would be very instructive to have a comprehensive study of the absolute accuracy of state-of-the-art radiative opacity calculations. A detailed comparison of the calculated opacity near the base of the CZ obtained by the Opacity Project (Seaton et al. 1994) with the results of the OPAL project (Iglesias & Rogers 1996) would be very informative. The interested reader is referred to the informative and insightful comparison by Neuforge-Verheecke et al. (2001) of the Los Alamos LEDCOP opacities and the OPAL opacities. The largest differences are found near the base of the CZ, with the OPAL opacities being as much as 6% larger than the LEDCOP opacities in this region. As part of a comprehensive discussion of factors that affect the accuracy of solar models, Boothroyd & Sackmann (2003) have investigated ways that the opacities can affect helioseismological parameters.

We show in § 3 that the radiative opacity near the base of the CZ depends sensitively on the assumed chemical composition (see especially eqs. [9] and [10]). If one wanted to calculate the depth to an accuracy of 0.6%, then one would need to know the heavy-element mass fraction,  $Z$ , to an accuracy of 1%. This precision is far beyond the current state-of-the-art accuracy in the determination of the heavy-element abundance.

The entire difference between the measured depth of the solar CZ (eq. [1]) and the value calculated using a solar model with the recent low determinations of the heavy-element abundances (eq. [2]) could be explained by the present uncertainty,  $\sim 15\%$ , in the ratio of  $Z/X$  (see Bahcall & Pinsonneault 2004). Of course, the changes in opacity caused by changing  $Z/X$  are not limited to any particular region. Changing the assumed surface value of  $Z/X$  affects the composition and hence the opacity throughout the solar model.

We have approximated in this paper the dependence of the opacity on composition by the dependence on just two variables, the mass fractions  $X$  and  $Z$ . In reality, the situation is more complex. Different chemical elements contribute differently to the stellar opacity. For example, Bahcall et al. (2001) found that the depth of the CZ was most sensitive to the abundances

of the lighter metals, which are significant opacity sources at  $2 \times 10^6$  K, while the heavier metals were much more important for the core structure and the estimated initial solar helium abundance. However, we are not yet at a level of precision that we can specify well the opacity-weighted uncertainties of the different heavy elements. This is a refinement that will have to await further progress in determining the different heavy-element abundances and more extensive opacity calculations.

We compare in § 4 the radiative opacity values obtained with two different interpolation routines from the standard OPAL opacity tables. We find that the difference in interpolated values of the radiative opacity can be as large as 4% near the base of the CZ. We also tested in § 5 the accuracy with which interpolations can be performed within simulated opacity tables of different grid sizes. We find that errors of the order of 3% may be expected from tables with the grid spacings of the existing OPAL tables. However, we show that the interpolation uncertainties could be reduced to the level of 1% or below by using a denser grid with  $\Delta \log T = 0.025$ ,  $\Delta \log r = 0.125$ , and with  $Z$  ranging from  $Z = 0.0100$  to  $0.0225$  with  $\Delta Z = 0.0025$ .

For completeness, we report in Appendix A on the calculated depth of the CZ that was found using four different EOSs. In agreement with other authors, we find that the choice of EOS affects the calculated depth of the CZ by only about  $\pm 0.1\%$ . We also show in Appendix A that current uncertainties in nuclear reaction rates also affect the calculated depth of the CZ at the level of 0.1%.

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## APPENDIX A

### IS THE EQUATION OF STATE THE CULPRIT?

In order to estimate the influence of the EOS on the calculated depth of the CZ, we have evolved a series of solar models using different EOSs. In addition to the OPAL 1996 EOS, we have used an updated version of the OPAL EOS (OPAL 2001; Rogers 2001), the MHD EOS (Mihalas et al. 1988), and the IRWIN EOS (Cassisi et al. 2003).

Table 2 summarizes our results. The variation in the calculated depth of the CZ due to varying the assumed EOS is

$$\frac{\Delta R_{CZ}}{R_{CZ}} \simeq 0.001. \quad (\text{A1})$$

This variation is similar to the quoted uncertainty in the measured depth of the CZ (see eq. [1]), but much smaller than the change in the calculated  $R_{CZ}$  required to obtain consistency with the new, lower heavy-element abundances (see eq. [2]). Similar results

TABLE 2  
DEPTH OF THE CONVECTIVE ZONE FOR DIFFERENT EOSs

EOS	$R_{bce}/R_{\odot}$
OPAL 1996 .....	0.7155
OPAL 2001 .....	0.7157
MHD .....	0.7164
IRWIN .....	0.7146

have been found previously by other authors (Schlattel 2002; Basu et al. 1999), who used, however, the larger value of  $(Z/X)_{\odot} = 0.0245$ .

We therefore conclude that the pressure-temperature-density relationship from the EOS is not a major component of the overall error budget for the depth of the solar surface CZ. However, the ionization balance of heavy elements as a function of the physical conditions can have a significant impact on the opacities; in this indirect sense, the EOS will have an impact on the problem.

## APPENDIX B

### HOW MUCH OF AN EFFECT DO NUCLEAR REACTIONS HAVE ON THE CALCULATED DEPTH OF THE CONVECTIVE ZONE?

For completeness, we record here the small effect that the rates of nuclear reactions have on the calculated depth of the solar CZ. In Table 1 of Bahcall & Pinsonneault (2004), the neutrino fluxes are listed for two models, BP00 and New Nuclear, that differ only in the adopted nuclear reactions. The New Nuclear model was computed using the best-estimate nuclear rates as of the end of 2003, while the model BP00 was computed using the best rates available in 2000. The computed depths for the CZ are 0.7140 (for BP00) and 0.7147  $R_{\odot}$  (for New Nuclear). Thus, the current uncertainties in the nuclear reaction rates affect the calculated depth of the solar CZ at the level of 0.1%.

## APPENDIX C

### THE CONVERSION OF CARBON AND OXYGEN TO NITROGEN DURING CNO BURNING

During the course of CNO burning, nearly all of  $^{12}\text{C}$  and a fraction of  $^{16}\text{O}$  are converted to  $^{14}\text{N}$  (for the earliest discussion of this process, see § II.C.2 of Bahcall & Ulrich 1988 and also § III.A of Bahcall & Pinsonneault 1992). This process increases slightly (decreases slightly) the heavy-element (hydrogen) mass fraction since, for example, two protons are added to  $^{12}\text{C}$  to make  $^{14}\text{N}$ .

Unfortunately, the enhancement of  $^{14}\text{N}$  at the expenses of hydrogen cannot be exactly taken into account with the existing OPAL opacity tables. The existing tables do not allow the selective enhancement of nitrogen.

We have therefore evolved two different solar models with two different treatments of the  $^{14}\text{N}$  enhancement. In the first model, the enhancement is taken into account and absorbed into the total heavy-element abundance,  $Z$ . This treatment correctly accounts for the increase in  $Z$  and the decrease in  $X$  when calculating the opacities but, incorrectly, spreads the increased heavy-element abundance among all of the metals according to their initial relative abundances. Thus, the solar interior opacity is slightly overestimated. In the second model, we completely ignored the increase in  $Z$  owing to the conversion of carbon and oxygen into nitrogen when computing the opacities. In this case, the solar interior opacity is slightly underestimated.

Fortunately, the fractional difference is only 0.1% for the computed depth of the solar CZ obtained with these two different approximations.

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