Finesse Enhancement Factors

Steve Adler, IAS, 12/29/06; expanded 11/4/08

First, the definition of finesse. From "Encyclopedia of Laser Physics and Technology", article on "Finesse", http://www.rp-photonics.com/finesse.html, one has $F \simeq 2\pi/(1-\rho)$, with $1-\rho$ the round trip power or intensity loss. From Wikipedia, article on "Fabry-Pérot Interferometer", http://en.wikipedia.org/wiki/Fabry-Perot" (with thanks to Will Happer for bringing this to my attention), one has $F \simeq \pi R^{\frac{1}{2}}/(1-R)$, with R the intensity reduction factor for a single mirror reflection. Writing $R = 1-\delta$, this becomes $F \simeq \pi/\delta$ to leading order. A round trip starting from any interior point in the interferometer requires two mirror reflections, hence the round trip fractional power loss is 2δ, and so $F = 2\pi/(2\delta)$, in agreement with the first on-line source quoted. Born and Wolf, 7th ed. [combining Eq. (16) on page 363 with Eq. (22) on page 365] defines finesse as $F \simeq \pi R^{\frac{1}{2}}/(1-R)$, which agrees with the Wikipedia definition. So all these sources agree on the definition of finesse.

Enhancement of transmitted amplitude reduction

Let $1-a$ be the amplitude reduction from a single pass. At each reflection, the reflected amplitude is reduced by $\sqrt{R}$, and the transmitted amplitude is reduced by $\sqrt{T}$, with $R+T = 1$. Referring to the Wikipedia article discussion of Fabry-Pérot transmission, and putting in the effect of absorption within the interferometer, one has for the transmitted amplitude

$$\begin{aligned}\text{transmitted amplitude} =& T(1-a)(1+R(1-a)^2+R^2(1-a)^4+...) \simeq \frac{(1-R)(1-a)}{1-R(1-2a)}\\ =& \frac{1-a}{1+[2R/(1-R)]a} \simeq 1-a(1+R)/(1-R) \simeq 1-(2F/\pi)a\end{aligned}$$

and so the single pass absorption coefficient a is enhanced by a factor $2F/\pi$, as asserted by the PVLAS group in analyzing vacuum dichroism.

Enhancement of transmitted amplitude phase change

Let $e^{i\Delta}$ be the phase change from a single pass. Then the total transmitted amplitude is

$$Te^{i\Delta}(1 + Re^{2i\Delta} + R^2 e^{4i\Delta} + ...)$$

This is just like the previous case with $a \to -i\Delta$, so the transmitted amplitude is

$$1 + (2F/\pi)i\Delta \simeq e^{i(2F/\pi)\Delta}$$

and so the phase change is enhanced by $2F/\pi$, as asserted by the PVLAS group in analyzing vacuum birefringence.

Calculation from the internal Fabry-Pérot field:

Polarization Parameters

Let us now consider the Fabry-Pérot etalon as operating in a steady state, and take account of the fact that at resonance, with an incident and exiting wave of amplitude unity, the internal field between the etalon mirrors consists of a right-moving wave and a left-moving wave with enhanced amplitudes. The right-moving wave has amplitude $1/T^{\frac{1}{2}}$, so that when multiplied by the transmission factor $T^{\frac{1}{2}}$ one gets the exiting wave amplitude of unity; an alternative way to see this is to form the sum of all the right-moving internal wave amplitudes, which is

$$T^{\frac{1}{2}}(1 + R + R^2 + ...) = T^{\frac{1}{2}}/(1 - R) = 1/T^{\frac{1}{2}}$$

The left-moving wave has the same amplitude (in the large finesse approximation $R^{\frac{1}{2}} \simeq 1$), but has the opposite phase from the left-moving externally reflected incident wave, so that the total external left-moving amplitude cancels to zero. (See Born and Wolf, 7th ed., pages

361-362 for details of the phases of the transmitted and reflected waves; the Wikipedia article glosses over some phase details, and assumes that $t = t'$ in the Born and Wolf discussion, so that the amplitude transmission factor is simply $T^{\frac{1}{2}}$.) Rewriting $1/T^{\frac{1}{2}}$ in terms of the finesse, we have

$$1/T^{\frac{1}{2}} = 1/(1-R)^{\frac{1}{2}} \simeq (F/\pi)^{\frac{1}{2}}$$

Since the $(F/\pi)^{\frac{1}{2}}$ amplitude enhancement is the same for both polarization states of the laser (parallel and perpendicular to the applied magnetic field), it has no effect on the ellipticity or the polarization axis orientation of the emerging laser beam, because the beam amplitude drops out in these quantities. Specifically, including the amplitude reduction a and phase $i\Delta$ discussed above, the emerging wave amplitude from the etalon is $1-(2F/\pi)(a-i\Delta)$, and the internal wave amplitude just before the exit mirror of the etalon is $(F/\pi)^{\frac{1}{2}}[1-(2F/\pi)(a-i\Delta)]$. The quantities a, Δ depend on the polarization state, while F/π is polarization independent; hence the ellipticity of the emerging beam and its polarization axis are the same, whether one calculates them with the amplitude outside the etalon, or using the enhanced amplitude inside the etalon.

Enhancement of scalar/pseudoscalar production:

fully coherent case

Consider now scalar or pseudoscalar particle production when a laser beam traverses a transverse magnetic field. Let σ be the rate for production of forward moving particles, in the absence of a Fabry-Pérot etalon. Now suppose that the magnetic field region is enclosed in an etalon, and let us neglect the small amplitude reduction a and small phase shift Δ introduced above, since the effect of these on particle production is second order small. For scalar or pseudoscalar production in the direction of the incident beam, the right-moving wave is relevant, and in the approximation of large finesse, its amplitude can be written as

$$1/T^{\frac{1}{2}} = 1/(1-R)^{\frac{1}{2}} \simeq (F/\pi)^{\frac{1}{2}}$$

Hence assuming complete coherence in the production process, the effect of the etalon is to enhance the production cross rate σ that would be obtained for a laser beam of unit amplitude by a factor of the amplitude enhancement squared, that is, by a factor F/π. In addition to forward production, the left-moving wave will produce left-moving particles with a rate $\sigma F/\pi$, and so the total particle production rate is σ times $2F/\pi$. Note that one does not obtain a finesse squared enhancement, because the amplitude inside the etalon is only enhanced by a factor scaling as the square root of the finesse.

Enhancement of scalar/pseudoscalar production by photon and photon production by scalar/pseudoscalar: multiple reflection calculation

Let us now do the calculation of the preceding section by the multiple reflection method. Consider first a laser beam incident on a magnetic field region contained within a Fabry-Perot interferometer, and let the photon to scalar/pseudoscalar conversion amplitude by the magnetic field be A. The photon beam has a transmission factor $T^{1/2}$ through the first mirror of the Fabry Perot, while the axion has an exiting transmission amplitude of 1, so ignoring the overall phase the direct amplitude is $AT^{1/2}$. However, the photon can also multiply reflect two, four, ... times before converting to a scalar/pseudoscalar, so the total exiting particle beam amplitude is

$$AT^{1/2}(1+R+R^2+...) = AT^{1/2}/(1-R) = A/T^{1/2} \simeq A(F/\pi)^{\frac{1}{2}}$$

For an incident scalar/pseudoscalar beam converting to a photon, the direct amplitude has an amplitude that is again $AT^{1/2}$, but there are also contributions where the produced photon reflects two, four, ... times before exiting, so the total exiting photon beam is again

$$AT^{1/2}(1+R+R^2+...) = AT^{1/2}/(1-R) = A/T^{1/2} \simeq A(F/\pi)^{\frac{1}{2}}$$

the same expression as for an incident photon (assuming the same conversion amplitude A in both cases). Note that in the incident photon case, the multiple reflections occur *before* conversion to the scalar/pseudoscalar, which then exits without further interaction with the Fabry-Perot, while in the incident scalar/pseudoscalar case, the multiple reflections occur *after* conversion to a photon, but the enhancement effect is the same. Squaring to get the enhancement of the intensity, one gets an intensity enhancement factor of F/π from each Fabry-Perot, or a factor $(F/\pi)^2$ when each magnetic field region has a Fabry-Perot, which are locked together in frequency, as in the Sikivie, Tanner, and van Bibber setup with matched Fabry-Perot interferometers. Note also that there is only one factor of A in the total amplitude, or of A^2 in the total intensity, so the formulas scale correctly to the case of very weak photon or particle beams, without further corrections. The amplitude here is a probability amplitude in the usual quantum mechanical sense, and the intensity a probability, and the formulas give the correct answer even when the probability is so low that events are infrequent.

Enhancement of scalar/pseudoscalar production:

partially coherent case

Now consider the alternative case, in which particle production is considered coherent over a single pass of the laser beam through the magnetic field, but multiple passes are not coherent with one another, so that one sums the rates (proportional to the amplitude squared) for each pass, instead of summing amplitudes and squaring afterwards, as in the fully coherent case. Let σ be the production rate associated with a single pass of the laser beam, in the absence of the etalon. When the etalon is introduced, the single pass amplitude is reduced by a factor $T^{\frac{1}{2}}$, so the single pass rate is reduced by a factor T. However, successive right-moving reflections contribute to the total forward production rate, and successive left-moving reflections contribue a backward production rate. For the total forward production rate we have

$$\sigma T(1 + R^2 + R^4 + ...) = \sigma(1 - R)/(1 - R^2) = \sigma/(1 + R) \simeq \frac{1}{2}\sigma$$

while for the total backward production rate we have

$$\sigma TR(1 + R^2 + R^4 + ...) = \sigma(1 - R)R/(1 - R^2) = \sigma R/(1 + R) \simeq \frac{1}{2}\sigma$$

So in this case the total production rate is σ, that is, there is no finesse enhancement, and the initial rate is now split between forward and backward production. The production rate in the fully coherent case, calculated above, is enhanced with respect to the production rate in the partially coherent case, by the factor $2F/\pi$. This is the same finesse enhancement factor that appeared above in considering the transmitted amplitude reduction and phase change.

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