1 Problem sets for dS/CFT (Maldacena)

1.1 Review of the two point function computation in dS.

Consider a massless scalar field in dS_4 with metric

$$ds^2 = \frac{-d\eta^2 + dx^2}{\eta^2} \tag{1}$$

a) Write the wave equation and solve it. Show that for each Fourier mode you get the solutions $f \sim (1 + i|k|\eta)e^{-i|k|\eta}$ and its complex conjugate.

b) Write $\phi = a^\dagger f + a f^\dagger$. Demand that $[a, a^\dagger] = 1$, use the canonical commutation relations for ϕ and π_ϕ and normalize f properly.

c) Define the Bunch Davies vacuum through $a|BD\rangle = 0$. Understand why this is a reasonable definition. Then compute

$$\langle BD|\phi_{\vec{k}}(\eta)\phi_{-\vec{k}}(\eta')|BD\rangle$$

d)** After defining this for each fourier mode, could you figure out how it looks once you sum over all Fourier modes?. Can you write the answer in a de Sitter invariant fashion ? Do you get a divergence?, why?

1.2 Interactions

Add a ϕ^3 interaction to the above problem and compute the three point function,

$$\langle BD|\phi_{\vec{k}_1}(\eta)\phi_{\vec{k}_2}(\eta)\phi_{\vec{k}_3}(\eta)|BD\rangle$$

Understand the Keldysh contour necessary for the computation. How do you kill off the oscillatory early time pieces ? Understand that you need to rotate the contour into the imaginary direction.

If you want a simpler problem, then consider a conformally coupled scalar field and add a ϕ^4 interaction. In this case the propagators are simpler and the computation is similar to the flat space one.

1.3 Direct computation of the wavefunction

a) Consider the action for the scalar field $S = \frac{1}{2}\int(\nabla\phi)^2$ in dS. Think about it in the flat slicing.

b) Compute the action with fixed boundary condition at some time η_c, and $\phi_b = \phi(\eta_c)$. At early times, put the positive frequency boundary condition $\phi \to e^{i|k|\eta}$.

c) Evaluate $e^{iS_{classical}}$ for the above solution.

d) Do the same in Euclidean AdS space $ds^2 = \frac{dz^2+dx^2}{z^2}$. Again put boundary conditions and evaluate the action. Understand the relation to c)

e) Compute the derivative of the wavefunctions with respect to the boundary conditions $\frac{\delta}{\delta\phi_b(\vec{k})}\frac{\delta}{\delta\phi_b(-\vec{k})}$ $e^{iS}|_{\phi_b=0}$. Can this be interpreted as the correlation function of a conformal field theory? Could you go to position space? Are there any IR divergencies in this case?

d) Understand why the Green's function

$$\langle 0|T\phi(\eta)\phi(\eta')|BD\rangle = f^*(\eta)f(\eta') - f(\eta)f(\eta') \quad , \quad 0 > \eta > \eta' \tag{2}$$

is the right one to use in the perturbative computation of the wavefunction. You could also try to normalize it correctly.

f)** You could attempt to write this answer in terms of proper distances in AdS. Namely, first go back to position space (don't forget the k dependent normalization factors) and write it in terms of proper distances. Does it have a singularity at the antipodal point?. What is the corresponding function in AdS space, and what is the antipotal point in this case.

1.4 Action for AdS_4

Compute the on shell action for AdS_4 for an S^3 boundary. Compute the action

$$S_{Euclidean} = \frac{R_{AdS}^2}{16\pi G_N}\left[-\int_{\Sigma_4}\sqrt{g}(R+6) - 2\int_{\partial\Sigma_4} K\right] \tag{3}$$

where K is the extrinsic curvature term, $K = \frac{1}{2}h^{ab}\partial_n h_{ab}$ where h is the metric of the boundary and n the normal direction. Write the AdS metric as

$$ds^2 = d\rho^2 + \sinh^2\rho d\Omega_3^2 \tag{4}$$

and evaluate the action as a function of the cutoff ρ_c. Discard the divergent terms and focus on the finite ones. Write down the final answer for $\Psi = Z \sim e^{-S_E}$.

1.5 Action for dS_4

Do the same for dS_4 with the Hartle-Hawking analytic continuation. Consider the metric

$$ds^2 = -d\tau^2 + \cosh^2\tau d\Omega_3^2 \tag{5}$$

Evaluate now $\Psi \sim e^{iS}$. Compare with the answer in the previous problem.

2 Exercises on Inflation (Creminelli)

1. Using symmetry arguments, calculate the tilt of the spectrum of a scalar with small mass, $m^2 \ll H^2$, in a fixed de Sitter background.

2. Using symmetry arguments, show that the n-point function of ζ in Fourier space in a generic model of inflation is of the form

$$\langle\zeta_{\vec{k}_1}\ldots\zeta_{\vec{k}_n}\rangle = (2\pi)^3\delta(\sum\vec{k}_i)F(k_i)\ , \tag{6}$$

   where F is an homogeneous function of the k's of degree $-3(n-1)$.

3. Calculate the equal time 2-point function of a massless scalar in a fixed de Sitter background in real space. What is the physical meaning of the IR divergence?

4. Photons are massless, but they are not produced during inflation. Why?

5. A Goldstone boson, like an axion, with decay constant f_a lives during inflation and therefore gets quantum perturbations. Estimate the corrections you expect to the free field theory calculation from self interactions, i.e. the deviation from gaussianity. What happens if $H > 4\pi f_a$?

6. Consider a massless scalar φ in de Sitter space with an interaction $\frac{M}{6}\varphi^3$. Calculate the 3-point function $\langle \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3} \rangle$.

7. de Sitter is maximally symmetric, i.e. it has 10 isometries. In inflation, the correlation functions of ζ are translationally, rotationally and scale invariant. These are 7 symmetries. What happened to the other 3?

8. Calculate (numerically) $f_{\rm NL}^{\rm loc.}$ in a model of modulated reheating coming from the non-linear relation $\zeta(\Gamma)$, as a function of Γ/H, where H is the Hubble parameter at the end of inflation.

3 Problems for Aspects of Eternal Inflation (Susskind)

1. Derive the metric of de Sitter space in the flat slicing.

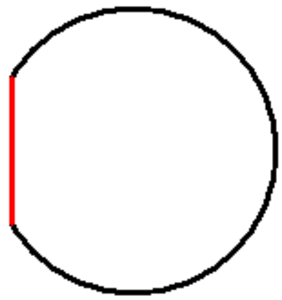
2. Derive the metric of de Sitter space in the static slicing.

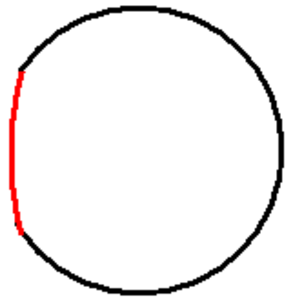
3. Prove that the spatial part of the metric in the static slicing corresponds to the metric of a 3 dimensional hemisphere.

4. Write down the rate equations for transitions between vacua. By using detailed balance and redefining variables show that the transition matrix can be made symmetric. Prove that the matrix has a zero eigenvalue and that all others are negative. Find the eigenvector with zero eigenvalue.

5. Consider the case of three vacua labeled $0, 1$ and 2. Assume 0 is terminal. Write down the transition matrix. Show that it has a zero eigenvalue and two negative ones. Find the eigenvector with zero eigenvalue. Show that the probabilities of finding the 1 and 2 vacua decrease with time but that the number of these vacua increases with time. Assume the elements of the transition matrix are small.

6. In the thin wall approximation the CDL instanton is constructed by gluing together two spaces along an infinitely thin seam. For ds $\to$ flat the spaces are a 4-sphere and a ball of flat space. The size of the ball is a free parameter but is should be less than the radius of the sphere. By analytic continuation find the corresponding Lorentzian signature metric in all regions I, II, III, IV, IV. For ds $\to$ ds the flat plane is replaced by a cap of a larger 4-sphere. Repeat the previous problem for this case. In the process you will discover a 4th slicing of de Sitter. (See figure)





4 Mechanisms for Inflation (E. Silverstein)

1) Consider a model of inflation based on a $\mu^{4-p}\lambda\phi^p$ potential (in the regime $\phi > M_P$), with $0 < p \leq 2$.

a) Determine the conditions on μ and the field range of ϕ during inflation to obtain N_e e-foldings of inflation, with a primordial power spectrum of the correct COBE-normalized magnitude over the e-foldings visible in the CMB.

b) Analyze the radiative stability of this model. Noting that, as discussed in the lectures, μ can arise as an exponentially small scale, discuss whether or not inflation over $N_e \geq 60$ e-foldings requires fine-tuning of the effective action from the Wilsonian point of view.

c) Discuss additional stability criteria which can arise in a UV completion of inflationary models (using string theory as a concrete example if you wish).

2) In this problem you will explore a few features of string compactifications and scalar fields that descend from them.

a) Consider compactifying a D-dimensional theory of gravity on a $D-4$ dimensional manifold X of linear size R. To be concrete, you can consider X to be a sphere, a torus, or compact hyperbolic space. This leads to a number of scalar fields in four dimensions, one of which corresponds to the volume V_X of X. By dimensionally reducing the Einstein term, show that the corresponding canonically normalized scalar field σ_X is given by

$$V_X = V_0 e^{c_X \sigma_X / M_P}$$

with c_X a number which is not parametrically small. If you feel energetic, include the string theory dilaton Φ and analyze the combined system of Φ and σ_X in the same way.

b) An important ingredient in string theory compactification is the *orientifold*. Without needing too much detail, this object sources a negative gravitational potential, but also introduces a Z_2 identification of space transverse to it (reflecting all the coordinates transverse to its extended directions). Discuss the difference between this and, say, a negative mass particle (or negative mass Schwarzchild solution) in terms of stability of the theory and validity of the second law of thermodynamics. (For the latter part, recall that the area of a black hole behaves like entropy.)

c) In the lectures we consider how axions $b = \int_\Sigma B$ work in string theory, with terms of the form $|dC_p + B \wedge dC_{p-2}|^2$ appearing in the effective action in addition to the kinetic energy $|dB|^2$. The latter term respects a gauge symmetry, $B \to B + d\Lambda_1$, where Λ_1 is a on-form gauge potential. Show that the former term also respects this symmetry, with an appropriate transformation law for C_p.

d) In the lectures we are also discussing brane motion inside compactifications. Consider for example D3-branes in a warped region. Consider an $AdS_5 \times X_5$ throat

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2 + ds^2_{X_5}$$

cutoff at $r = r_{UV}$ and connected to a compactification. A 3-brane can move in r, with canonically normalized field $\phi \sim r/\alpha'$. Compute the four-dimensional Planck mass as a function of the maximum value of the canonically normalized field $\phi_{UV} \sim r_{UV}/\alpha'$ and the geometry of the compact dimensions. Using the Lyth bound, convert this to a bound on the tensor to scalar ratio as a function of these quantities.

3) Derive the equations of motion for the homogeneous field and perturbations for inflation governed by action $S = -\int d^4x\sqrt{-g}[(\phi^4/\lambda)\sqrt{1-\lambda(\partial_\mu\phi\partial^\mu\phi)/\phi^4} - V(\phi)]$. Show explicitly that inflation occurs on a steep potential, one which does not satisfy the slow roll conditions.

4) Consider the model of warped D-brane slow-roll inflation discussed in the lectures. Estimate the tension of cosmic strings produced at the end (in suitable cases) as a function of the parameters of the model.

5 Robustness of GR. Attempts to modify gravity (Arkani-Hamed)

1. Let us verify that an action of the form

$$S = \int d^4x\sqrt{-g}F(R) \tag{7}$$

is equivalent to GR + a scalar with standard kinetic term. Show that the action above is equivalent to

$$\int d^4x\sqrt{-g}\left[F'(A)(R-A) + F(A)\right] \tag{8}$$

once the equation of motion for the auxiliary field A is used. Now we can make a conformal transformation to demix A from the metric; this will give a kinetic term to A. After dust has settled you should get

$$\int d^4x\sqrt{-g}\left[R - \frac{3}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V(\sigma)\right] \tag{9}$$

with

$$V(\sigma) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} \qquad \text{with } \sigma = -\log F'(A)\ . \tag{10}$$