Problem assignments by group 3

July 28, 2011

Susskind 3:

The metric of de Sitter space in static slicing is:

$$ds^2 = -(1 - \frac{r^2}{R^2})d\tau^2 + (1 - \frac{r^2}{R^2})^{-1}dr^2 + r^2 d\Omega_2^2 \tag{1}$$

Taking coordinate transformation: $r = R\sin\psi$ and take the constant time surface $\tau = constant$,we have the metric in the form:

$$ds^2 = R^2 d\psi^2 + R^2 sin^2\psi d\Omega_2^2 \tag{2}$$

It is the metric of a three-dimensional hemisphere, it is not a sphere since for ψ in the range of $[\frac{\pi}{2}, \pi]$, same values for r would be obtained and therefore identified with the ψ within $[0, \frac{\pi}{2}]$

Creminelli 4:

The equation of motion for the gauge field under consideration is conformally invariant. A conformally invariant theory defined on a conformally flat spacetime means that the dynamics and behaviour of the field is as if it were in a Minkowski background. As such photon production cannot take place during inflation for the same reasons that we get no photon production in a Minkowski spacetime. We would need to introduce a source or interaction for particle production. Some extensions that could be considered are: 1) direct coupling between the photon and inflaton. 2) Metric perturbations that couple universally to both the inflaton and photon field.

Maldacena 3:

Given a massless scalar field in a de Sitter background the general solutions for the mode functions were shown to be:

$$\phi_k = A_k (1 - ik\eta) e^{ik\eta} + B_k (1 + ik\eta) e^{-ik\eta}, \tag{3}$$

The appropriate initial condition was to isolate the positive frequency modes in the limit $\eta \to -\infty$ such that $\phi \to e^{ik\eta}$. The flat slicing of de Sitter space can be written as:

$$ds^2 = \eta^{-2} \left(-d\eta^2 + \delta_{ij} dx^i dx^j\right); \qquad g_{\mu\nu} = \eta^{-2}\eta_{\mu\nu}; \qquad g^{\mu\nu} = \eta^2\eta^{\mu\nu}, \tag{4}$$

with $\eta_{\mu\nu}$ being the Minkowski spacetime metric written in conformal time. The action for a massless scalar field was given by:

$$
\begin{aligned}
S &= \frac{1}{2}\int d^3x d\eta \sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \\
&= \frac{1}{2}\int d^3x d\eta\, \eta^{-4}\left[-\eta^2\left(\partial_\eta\phi\right)^2 + \eta^2\left(\nabla\phi\right)^2\right], \\
&= \frac{1}{2}\int d^3x d\eta\, \eta^{-2}\left[\left(\partial_\eta\phi(x)\right)\left(\partial_\eta\phi(x')\right) + \nabla\phi(x)\nabla\phi(x')\right].
\end{aligned} \tag{5}
$$

The procedure we will adopt is to compute the action for fixed boundary conditions at some time η_c such that $\phi^b = \phi(\eta_c)$ and use this boundary condition to evaluate the partition function for the classical action, $Z[\phi] = e^{iS_{cl}}$.

Evaluating the mode functions at the boundary condition and using this to normalise the mode functions yields:

$$
\begin{aligned}
\frac{\phi_k}{\phi_k^b} &= \frac{(1-ik\eta)e^{ik\eta}}{(1-ik\eta_c)e^{ik\eta_c}}, \\
\phi_k &= \phi_k^b\frac{(1-ik\eta)e^{ik\eta}}{(1-ik\eta_c)e^{ik\eta_c}}.
\end{aligned} \tag{6}
$$

We can expand the scalar field in the action above in terms of the modes ϕ_k by:

$$
\phi(x) = \int \frac{d^3k}{(2\pi)^3}\phi_k(\eta)e^{ikx}, \tag{7}
$$

such that the action reduces to:

$$
S = \frac{1}{2}\int d^3x d\eta\eta^{-2}\frac{d^3k}{(2\pi)^3}e^{ix(k+k')}\left\{\partial_\eta\phi_k\partial_\eta\phi_{k'} - kk'\phi_k\phi_{k'}\right\}, \tag{8}
$$

$$
= \frac{1}{2}\int \frac{d\eta}{\eta^2}\frac{d^3k}{(2\pi)^3}\left\{\partial\phi_k\partial\phi_{-k} + k^2\phi_k\phi_{-k}\right\}. \tag{9}
$$

We can then perform an integration by parts noting that if we use the equations of motion for the scalar field then the terms proportional to k^2, the bulk terms, will drop out leaving only a boundary contribution. We neglect terms that have an oscillatory dependence. The result is:

$$
S = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\eta_c^2}\left\{\partial_\eta\phi_k\Big|_{\eta=\eta_c}\phi_{-k}^b\right\}. \tag{10}
$$

Given that,

$$\partial_\eta \phi_k \Big|_{\eta=\eta_c} = \phi_k^b \frac{k^2 \eta_c}{(1 - ik\eta_c)} \tag{11}$$

we find

$$S = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \phi_k^b \phi_{-k}^b \frac{k^2}{\eta_c (1 - ik\eta_c)} \tag{12}$$

Series expanding the bracket gives

$$\begin{aligned} S &= \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \phi_k^b \phi_{-k}^b \frac{k^2}{\eta_c} \left[1 + ik\eta_c + \mathcal{O}(k^2)\right], \\ &= \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \phi_k^b \phi_{-k}^b \left\{ \frac{k^2}{\eta_c} + ik^3 + \mathcal{O}(k^4) \right\}. \end{aligned} \tag{13}$$

In order to calculate an N-point correlator we associate our classical action to a partition function which will act as the correlator generating function. We assume that the wavefunction is approximated by the classical action in a semiclassical limit: $\Psi_{dS} \approx e^{iS_{cl}}$. The partition function is given by:

$$\Psi_{dS} = Z_{CFT}[\phi_{\partial dS}] \approx e^{iS_{cl}[\phi]} = \left\langle \exp\left(\int_{\partial dS} d^3x \phi \mathcal{O} \right) \right\rangle. \tag{14}$$

Hence the N-point correlators are generated by:

$$\frac{\delta^n Z[\phi]}{\delta\phi_1 \dots \delta\phi_n} \Bigg|_{\phi=0} = \langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle. \tag{15}$$

This means that the two-point correlation function is given by:

$$\frac{\delta^2 Z[\phi]}{\delta\phi_k^b \delta\phi_{-k}^b} \Bigg|_{\phi_k^b=0} = \frac{1}{(2\pi)^3} \left\{ i\frac{k^2}{\eta_c} - k^3 + \mathcal{O}(k^4) \right\} = \langle \phi\phi \rangle. \tag{16}$$

The first term can be viewed as a divergent term in η_c.

A similiar approach can be taken for Euclidean AdS space with metric

$$ds^2 = z^{-2}\left(dz^2 + \delta_{ij} dx^i dx^j\right). \tag{17}$$

The action for a massless scalar in this background is given by:

$$S = \frac{1}{2} \int d^3x \frac{dz}{z^2} \left[(\partial_z \phi)^2 + (\nabla\phi)^2 \right]. \tag{18}$$

Using the Bunch-Davies prescription and normalising at a given boundary condition we find that the appropriate solutions are

$$\phi_k = \phi_k^b \frac{(1+kz)e^{-kz}}{(1+kz_c)e^{-kz_c}}. \tag{19}$$

Expanding the field in terms of its modes and substituting into the action gives

$$S = \frac{1}{2}\int d^3x \frac{dz}{z^2}\frac{d^3k}{(2\pi)^3} e^{ix(k+k')}\left[\partial_z\phi_k\partial_z\phi_{k'} - kk'\phi_k\phi_{k'}\right], \tag{20}$$

$$= \frac{1}{2}\int \frac{dz}{z^2}\frac{d^3k}{(2\pi)^3}\left[\partial_z\phi_k\partial_z\phi_{-k} + k^2\phi_k\phi_{-k}\right]. \tag{21}$$

Again we integrate by parts to reduce the action to a boundary contribution,

$$S = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{z_c^2}\left[\phi_{-k}^b \frac{d\phi_k}{dz}\bigg|_{z=z_c}\right]. \tag{22}$$

Given that

$$\frac{d\phi_k}{dz}\bigg|_{z=z_c} = \phi_k^b \frac{k^2 z_c}{(1+kz_c)}. \tag{23}$$

Series expanding

$$S = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{z_c^2}\phi_k^b\phi_{-k}^b \frac{k^2}{z_c(1+kz_c)}, \tag{24}$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\phi_k^b\phi_{-k}^b\left\{-\frac{k^2}{z_c} + k^3 + \mathcal{O}(k^4)\right\}.$$

Taking the functional derivative of the partition function with respect to the fields we can derive the 2-point correlator as before,

$$\frac{\delta^2 Z[\phi]}{\delta\phi_k^b\delta\phi_{-k}^b}\bigg|_{\phi_k^b=0} = \frac{1}{(2\pi)^3}\left(-\frac{k^2}{z_c} + k^3\right) = \langle\phi\phi\rangle. \tag{25}$$

The term proportional to k^2 can be viewed as a divergent in the CFT. This expression for the 2-point correlator and the previous expression for the 2-point correlator can be related by letting $\eta \to iz$ with an equivalence up to a sign as a consequence of this fact.

The appropriate Green's function to use in the perturbative calculation of the wavefunction can be expressed by

$$\langle 0|T\phi(\eta)\phi(\eta')|BD\rangle. \tag{26}$$

This function sets the Bunch-Davies vacuum as our initial state. The initial vacuum state is then allowed to evolve under a time-varying gravitational field in the form of an expanding spacetime. As a result of the time varying nature of the gravitational field the vacuum state defined at later times will not be equivalent to the initial vacuum state due to resulting particle production in a process that may be thought of as a type of Schwinger effect. As a result there will be a transition amplitude associated with the wavefunction evaluated at some arbitrary time with an instantaneously defined vacuum state. Time ordering becomes important when considering a time-dependent system with a time-dependent vacuum.

Silverstein 2: b):

For the negative mass particle, it generates a Schwarzchild solution with negative mass parameter around it, such a geometry doesn't have an event horizon, only a bare singularity with a negative ADM mass. On one hand, we could create such particles at rest indefinitely so that the total energy could be lowered indefinitely, on the other hand such a geometry itself is not stable under perturbation. But for orientifold, since its non-local nature, creating such a configuration requires a non-trivial change in the boundary condition far away, therefore it is hard to create them indefinitely, on the other hand, the geometry itself is stable against the perturbations that are even in space—because of the constraint of mirror symmetry.

Notation: Indices at the start of the greek alphabet are used to denote all D dimensions $0,\ldots,D-1$. Indices $\mu,\nu,\ldots$ are used for four-dimensional sums over $0,1,2,3$, while roman letters $I,J,\ldots$ are used for the compactified dimensions $4,\ldots,D-1$. Let us assume that the metric has a block-diagonal form, Where the 4-metric may have an arbitrary dependence on the macroscopic dimensions, the metric on the compactification manifold on the other hand only varies by an overall length scale.

$$g_{\alpha\beta} = \begin{pmatrix} g^{(4)}_{\alpha\beta}(x_\mu) & 0 \\ 0 & e^{\sigma(x_\mu)}\gamma_{\alpha\beta} \end{pmatrix} \tag{27}$$

This gives us the following nonzero Christoffel symbols with mixed indices (both $\mu,\nu,\cdots$ and $I,J,\ldots$). The rank-1 Christoffel symbols need not be calculated.

$$\Gamma^{\rho}_{IJ} = -(\nabla^{\rho}\sigma)e^{\sigma(x)}\gamma_{IJ} \tag{28}$$

$$\Gamma^{J}_{\mu I} = (\nabla_{\mu}\sigma)\delta^{J}_{I} \tag{29}$$

Now we will write out the definition of the Ricci scalar, splittig the summations into μ,ν

and I, J ranges.

$$
\begin{aligned}
R = R^{(4)} + R^{(\gamma)} e^{-\sigma(x)} + \\
g^{\mu\nu}\big(\partial_I \Gamma \mu\nu^I - \underline{\partial_\nu \Gamma^I_{\mu I}} + \Gamma^I_{\mu\nu}\Gamma^\alpha_{I\alpha} + \underline{\Gamma^\rho_{\mu\nu}\Gamma^I_{\rho I}} - \Gamma^\rho_{\mu I}\Gamma^I_{\nu\rho} - \Gamma^J_{\mu\rho}\Gamma^\rho_{\nu I} - \underline{\Gamma^J_{\mu I}\Gamma^I_{\nu J}}\big) \\
+ e^{-\sigma(x)}\gamma^{IJ}\big(\underline{\partial_\mu \Gamma^\mu_{IJ}} - \partial_J \Gamma^\mu_{I\mu} + \underline{\Gamma^\mu_{IJ}\Gamma^\nu_{\mu\nu}} + \Gamma^K_{IJ}\Gamma^\nu_{K\nu} + \\
\underline{\Gamma^\mu_{IJ}\Gamma^K_{\mu K}} - \Gamma^\nu_{I\mu}\Gamma^\mu_{J\nu} - \underline{\Gamma^K_{I\mu}\Gamma J K^\mu} - \underline{\Gamma^\mu_{IK}\Gamma^K_{J\mu}}\big) \quad (30)
\end{aligned}
$$

where we have collected terms with only one type of index into the Ricci scalar for the 4-metric $R^{(4)}$ and the curvature scalar for the compactification manifold $R^{(\gamma)}$. Only the underlined terms are nonvanishing and evaluate to

$$
\begin{aligned}
R^{(4)} + R^{(\gamma)} e^{-\sigma(x)} - (D-4) g^{\mu\nu}\partial_\nu \nabla_\mu \sigma + (D-4)\Gamma^\rho_{\nu\mu}\nabla_\rho \sigma - (D-4)(\nabla_\mu\sigma)(\nabla^\mu\sigma) \\
+ (-1)(D-4)\big(\partial\nabla^\mu\sigma + (\nabla_\mu\sigma)(\nabla^\mu\sigma)\big) + (-1)(D-4)\Gamma^\mu_{\mu\rho}\nabla^\rho\sigma \\
+ (-1)(D-4)^2(\nabla_\mu\sigma)(\nabla^\mu\sigma) - (-1)(D-4)(\nabla_\mu\sigma)(\nabla^\mu\sigma) - (-1)(D-4)(\nabla_\mu\sigma)(\nabla^\mu\sigma) \quad (31)
\end{aligned}
$$

where we can combine the terms such that we get an expression with covariant derivatives

$$
R^{(4)} + R^{(\gamma)} e^{-\sigma(x)} - 2(D-4)\nabla_\mu\nabla^\mu\sigma - (D-4)^2(\nabla_\mu\sigma)(\nabla^\mu\sigma) \quad (32)
$$

In the action

$$
\int d^D x \sqrt{-g} R = \int d^{(D-4)} x \sqrt{\gamma} \int d^4 x \sqrt{-g^{(4)}} e^{(D-4)\sigma/2}\big(-2(D-4)\nabla_\mu\nabla^\mu\sigma + \dots\big) \quad (33)
$$

we can partially integrate the term with the double-derivative and get

$$
\int d^4 x \sqrt{-g^{(4)}} e^{(D-4)\sigma/2}\big(V_0 R^{(4)} + I e^{-\sigma(x)}\big) \quad (34)
$$

where the $(\nabla\sigma)^2$ term cancels. The constant I is

$$
I = \int d^{(D-4)} x \sqrt{\gamma} R^{(\gamma)} \quad (35)
$$

In this formulation however, σ appears like a dilaton. "Changing frame", i.e. making an appropriate conformal transformation, to get rid of this

$$
g^{(4)}_{\mu\nu} \to \tilde{g}^{(4)}_{\mu\nu} = g^{(4)}_{\mu\nu} \frac{1}{V_0} e^{-(D-4)\sigma/2} \quad (36)
$$

leads to the following form of the action

$$
\int d^4 x \sqrt{-\tilde{g}^{(4)}} \left(\tilde{R}^{(4)} - \frac{3}{2}\left(\frac{D-4}{2}\right)^2 (\nabla^\mu\sigma)(\nabla_\mu\sigma) + \frac{I}{V_0^2} e^{-(D-2)\sigma(x)/2} \right) \quad (37)
$$

where a total derivative term $\propto \nabla^\mu\nabla_\mu\sigma$ has been dropped. We can now canonically normalize σ

$$
\sigma \to \tilde{\sigma} = \sqrt{3}\frac{D-4}{2} \quad (38)
$$

and get the final action

$$\int d^4x\sqrt{-\tilde{g}^{(4)}}\left(\tilde{R}^{(4)} - \frac{1}{2}(\nabla^\mu\tilde{\sigma})(\nabla_\mu\tilde{\sigma}) + \frac{I}{V_0^2}e^{-\frac{D-2}{2\sqrt{3}(D-4)}\tilde{\sigma}(x)}\right) \tag{39}$$

For the case of superstring theory $D = 10$, the coefficient in the exponent reads $\frac{2}{3\sqrt{3}}$.

Silverstein 2: c)

The gauge transformation

$$B \to B + d\Lambda \tag{40}$$

leaves dB invariant.

$$dB \to dB + dd\Lambda = dB \tag{41}$$

In order for the (p+1)-Form

$$\tilde{F}_{p+1} = dC_p + B \wedge dC_{p-2} \tag{42}$$

to be invariant as well, C_p has to transform under Λ as:

$$C_p \to -\Lambda \wedge dC_{p-2} \tag{43}$$

Then

$$\begin{aligned}\tilde{F}_{p+1} &\to d(C_p - \Lambda \wedge dC_{p-2}) + (B + \Lambda) \wedge dC_{p-2} \\ &= dC_p - d\Lambda \wedge dC_{p-2} + B \wedge dC_{p-2} + d\Lambda \wedge dC_{p-2} \\ &= dC_p + B \wedge dC_{p-2}\end{aligned} \tag{44}$$