Maldacena #2 **Group 23**

$$\langle BD|\phi_{\vec k_1}(\eta)\phi_{\vec k_2}(\eta)\phi_{\vec k_3}(\eta)|BD\rangle \approx \langle 0|U^{-1}\phi_{\vec k_1}(\eta)\phi_{\vec k_2}(\eta)\phi_{\vec k_3}(\eta)U|0\rangle$$

$$U|0\rangle = T\int_{-\infty}^{\eta} e^{iH_{int}(\eta')}d\eta', \text{ where } H_{int}(\eta') = g\int d^3x \frac{1}{H^4\eta'^4}\phi^3(x,\eta').$$

metric det from $ds^2 = \frac{-d\eta^2 + d\vec x^2}{H^2\eta^2}$

To first order in g, we have

$$ig\langle 0|\left[\phi_{\vec k_1}(\eta)\phi_{\vec k_2}(\eta)\phi_{\vec k_3}(\eta), \int_{-\infty}^{\eta} d\eta' d^3x\, d^3x'\, d^3x''\, \frac{1}{H^4\eta'^4} e^{i\vec k_4\cdot\vec x + i\vec k_5\cdot\vec x' + i\vec k_6\cdot\vec x''}\phi_{\vec k_4}(\eta')\phi_{\vec k_5}(\eta')\phi_{\vec k_6}(\eta')\right]|0\rangle$$

where time-ordering does not matter to this order.

$\langle \phi_{\vec k_1}(\eta)\phi_{\vec k_2}(\eta)\phi_{\vec k_3}(\eta)\phi_{\vec k_4}(\eta')\phi_{\vec k_5}(\eta')\phi_{\vec k_6}(\eta')\rangle$ is symmetric in $\{1\,2\,3\}$ and in $\{4\,5\,6\}$.

We use Wick's theorem to decompose the correlator into contractions, where

$$\phi_{\vec k}(x)\phi_{\vec k'}(y) = f^*_{\vec k}(x)f_{\vec k}(y)(2\pi)^3\delta^3(\vec k+\vec k'), \text{ and } f_{\vec k}(x) = \frac{H}{\sqrt{2k^3}}(1+ikx)e^{-ikx}$$

From the symmetry, we need only consider

1 2 3 4 5 6 $\rightarrow$ proportional to $\delta^{(3)}(\vec k_3)$; does not contribute to 3-pt function!

1 2 3 4 5 6 There is a symmetry factor of 6.

$$= -6i\lambda\, \mathrm{Im}\int_{-\infty}^{\eta} d\eta'(2\pi)^3\delta^3(\vec k_1+\vec k_2+\vec k_3)\frac{1}{H^4\eta'^4}\prod_{i=1}^{3} f^*_{\vec k_i}(\eta)f_{-\vec k_i}(\eta')]$$

$$= -6i\lambda(2\pi)^3\delta^3(\vec k_1+\vec k_2+\vec k_3)\frac{1}{H^4}\frac{H^6}{\prod_i(2k_i^3)}\,\mathrm{Im}\int_{-\infty}^{\eta} d\eta'\frac{1}{(\eta')^4}e^{-i(k_1+k_2+k_3)(\eta'-\eta)}\prod_i(1-ik_i(\eta-\eta')+k_i^2\eta\eta')$$

$$= -\frac{3}{4}i(2\pi)^3\delta^3(\vec k_1+\vec k_2+\vec k_3)\frac{H^2}{k_1^3k_2^3k_3^3}\left[\mathrm{Im}\int_{-\infty(1-i\epsilon)}^{\eta}\frac{d\eta'}{\eta'^4}e^{-i(k_1+k_2+k_3)(\eta'-\eta)}\prod_i(1-ik_i(\eta-\eta')+k_i^2\eta\eta')\right]$$

where the $i\epsilon$ prescription projects onto the Minkowski vacuum at early times.

At late times, we can take $k\eta\to 0$, and then evaluate the integral on $(-\infty(1-i\epsilon), \eta_0)$. The $\frac{1}{\eta_0^3}$, $\frac{1}{\eta_0^2}$ terms are real, and the leftover piece is

$$\left[-\frac{1}{9}(-4+3\gamma+3\log(-\eta)+3\log(k_1+k_2+k_3))(k_1^3+k_2^3+k_3^3) + \frac{1}{3}\left(\sum_{i\neq j}k_ik_j^2 - k_1k_2k_3\right)\right] \text{ where } \gamma\approx 0.577\ldots$$

So the leading contribution at late times is given by

$$\boxed{-\frac{i}{4}(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{H^2}{k_1^3k_2^3k_3^3}\left\{-\frac{1}{3}\Big(-4+3\gamma+\overbrace{3\log(-\eta_0)+3\log(k_1+k_2+k_3)}^{\text{log divergence}}\Big)(k_1^3+k_2^3+k_3^3) + \Big(\sum_{i\neq j}k_ik_j^2-k_1k_2k_3\Big)\right\}}$$

Thank you Mathematica..

2.1 (group 23)

$$S = -\frac{1}{2}\int \sqrt{-g}\,\{(\partial\phi)^2 + m^2\phi^2\}$$

$$ds^2 = \frac{-d\eta^2 + d\vec{x}^2}{\eta^2}$$

$$\Rightarrow S = \frac{1}{2}\int \frac{d^3x\, d\eta}{H^2\eta^2}\left\{(\partial_\eta\phi)^2 - \partial_i\phi\,\partial_i\phi - \frac{m^2}{H^2\eta^2}\phi^2\right\}$$

invariant under $\begin{cases}\eta \to \lambda\eta \\ \vec{x} \to \lambda\vec{x}\end{cases} \Rightarrow \vec{k} \to \lambda^{-1}\vec{k}$

$$\phi(\vec{x},\eta) = \int d^3k\, \phi_{\vec{k}}(\eta)\, e^{i\vec{k}\cdot\vec{x}} \Rightarrow \phi_{\vec{k}} \to \lambda^3\phi_{\lambda\vec{k}}$$

$$\Rightarrow \langle \phi_{\vec{k}}\, \phi_{\vec{k}'}\rangle \sim \underbrace{(2\pi)^3\delta(\vec{k}+\vec{k}')}_{\text{FROM TRANSLATIONAL INVARIANCE}}\ \underbrace{\frac{f(k\eta, \frac{m}{H})}{k^3}}_{\text{FROM SCALING ARGUMENT}}$$

Let us determine the time dependence of $\phi_{\vec{k}}(\eta)$ in the $k \to 0$ limit

$$\frac{\delta S}{\delta\phi} = 0 \Rightarrow \phi'' - \underbrace{\nabla^2\phi}_{\substack{\to 0 \\ \text{when } k\to 0}} - \frac{2}{\eta}\phi' + \frac{m^2}{H^2\eta^2}\phi = 0$$

$$\phi'' - \frac{2}{\eta}\phi' + \frac{m^2}{H^2\eta^2}\phi \underset{k\to 0}{\approx} 0$$

ANSATZ: $\phi_{\vec{k}=0}(\eta) \sim \eta^\alpha \Rightarrow \alpha^2 - 3\alpha + \frac{m^2}{H^2} = 0$

$$\Rightarrow \boxed{\alpha_\pm = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - \frac{m^2}{H^2}}}$$

$$\Rightarrow \langle \phi_{\bar{k}} \phi_{\bar{k}'} \rangle \underset{k\to 0}{\sim} (k\eta)^{2\alpha_-} (2\pi)^3 \delta(\bar{k}+\bar{k}') \frac{1}{k^3} \overset{m \ll H}{\approx}$$

(the mode η^{α_+} "decays" faster because $\alpha_+ > \alpha_-$)

$$\approx (k\eta)^{2m^2/(3H^2)} (2\pi)^3 \delta(\bar{k}+\bar{k}') \frac{1}{k^3} \sim$$

$$\sim (2\pi)^3 \delta(\bar{k}+\bar{k}') \frac{k^{n_s - 1}}{k^3}$$

$$\Rightarrow \boxed{n_s = 1 + \frac{2m^2}{3H^2}}$$

Susskind #6

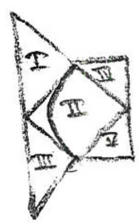
In the thin-wall approx, the CdL instanton for dS → flat looks like:

$$\frac{ds^2}{R_{dS}^2} = d\xi^2 + f(\xi)^2(d\alpha^2 + \sin^2\alpha\, d\Omega_{d-2}^2)$$

$$f(\xi) = \begin{cases} \xi & 0 \le \xi \le \sin\theta_0 \\ \sin(\xi + \theta_0 - \sin\theta_0), & \sin\theta_0 \le \xi \le \pi - \theta_0 + \sin\theta_0 \end{cases}$$

where $R_{dS}\sin\theta_0 \le R_{dS}$ is the radius of the ball of flat space.

Analytic continuation to Lorentzian signature:



I. $\xi \to it_I$, $\alpha \to ir$

$$\Rightarrow -dt_I^2 + a(t_I)^2(dr^2 + \sinh^2 r\, d\Omega_{d-2}^2)$$

III. $\xi \to -it_{III}$, $\alpha \to ir$

so that t_{III} increases as we go up the diagram.

II: $\alpha \to it_{II}$, $d\Omega_{d-2}^2 \to dH_{d-2}^2$

$$\Rightarrow d\xi^2 + f(\xi)^2(-dt_{II}^2 + \sinh^2 t_{II}\, dH_{d-2}^2)$$

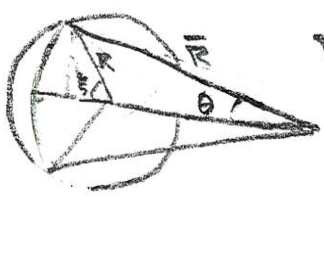
IV: $\xi \to it_{IV}$, $\alpha \to ir$

$$-dt_{IV}^2 + \sinh^2(t_{IV} - t_{IV,0})\underbrace{(dr^2 + \sinh^2 r\, d\Omega_{d-2}^2)}_{dH_{d-1}^2}$$

This is the "open slicing" of de Sitter.

V: $\xi \to -it_V$, $\alpha \to ir$.

dS → dS: the CdL instanton consists of a pair of spherical zones, w/ radius of curvature $R, \bar{R}$.



$R\sin\xi$ $\bar{R}\sin\theta$

metric here is $\bar{R}\,d\theta^2 + \bar{R}^2\sin^2\theta(d\alpha^2 + \sin^2\alpha\, d\Omega_{d-2}^2)$

$$ds^2 = \frac{R^2\cos^2\xi}{\left(1 - \frac{R^2}{\bar{R}^2}\sin^2\xi\right)}d\xi^2 + R^2\sin^2\xi(d\alpha^2 + \sin^2\alpha\, d\Omega_{d-2}^2)$$

for $0 \le \xi \le \sin\theta_0$.

The analytic continuation proceeds as before, since the relation $R \sin \xi = \bar{R} \sin \theta$ commutes with $\theta \to i\theta'$, $\xi \to i\xi'$.

$$\underline{4.3} \quad S = +\int \sqrt{-g}\left\{\frac{\phi^4}{\lambda}\sqrt{1-\lambda\partial_\mu\phi\partial^\mu\phi/\phi^4} - V(\phi)\right\}$$

$$\frac{\delta S}{\delta\phi} = 0 \Rightarrow \frac{4\phi^3}{\lambda}\sqrt{\quad} - V'(\phi) \qquad \underline{(\lambda<0)}$$

$$+\frac{1}{2}\left(\frac{4\lambda(\partial\phi)^2}{\phi^5} + 2\lambda\vec{\partial}\,\frac{\partial\phi}{\phi^4}\right)\frac{\phi^4}{\lambda}\left[1-\lambda\frac{(\partial\phi)^2}{\phi^4}\right]^{-1/2}$$

$$= \frac{4\phi^3}{\lambda}\sqrt{1-\lambda\frac{(\partial\phi)^2}{\phi^4}} - V'(\phi)$$

$$+\frac{2(\partial\phi)^2}{\phi}\left[1-\lambda\frac{(\partial\phi)^2}{\phi^4}\right]^{-1/2} +$$

$$+\Box\phi\left[\quad\right]^{-1/2} - \frac{\partial\phi}{2}\left[\quad\right]^{-3/2}(-\lambda)\partial\frac{(\partial\phi)^2}{\phi^4} =$$

$$= \frac{4\phi^3}{\lambda}\Omega - V'(\phi) + \left[\frac{2(\partial\phi)^2}{\phi} + \Box\phi\right]\Omega^{-1} +$$

$$+\lambda\Omega^{-3}\frac{\partial_\mu\phi}{\phi^4}\left[\partial^\mu\partial^\lambda\phi\,\partial_\lambda\phi - 2\frac{(\partial\phi)^2}{\phi}\partial^\mu\phi\right] = 0$$

Where $\Omega \equiv \sqrt{1-\lambda(\partial\phi)^2/\phi^4}$

<u>(group #23)</u>

$$\frac{\delta S}{\delta g^{\mu\nu}} = \sqrt{-g}\left\{-\frac{1}{2}g_{\mu\nu}\mathcal{L} + \frac{\phi^4}{2\lambda}\Omega^{-1}(-\lambda)\frac{\partial_\mu\phi\,\partial_\nu\phi}{\phi^4}\right\} =$$

$$= \sqrt{-g}\left\{-\frac{g_{\mu\nu}}{2}\left[\frac{\phi^4}{\lambda}\Omega - V(\phi)\right] - \frac{1}{2}\Omega^{-1}\partial_\mu\phi\,\partial_\nu\phi\right\} \equiv$$

$$\equiv -\frac{\sqrt{-g}}{2}T_{\mu\nu}$$

$$\Rightarrow T_{\mu\nu} = g_{\mu\nu}\left[\frac{\phi^4}{\lambda}\Omega - V\right] + \Omega^{-1}\partial_\mu\phi\,\partial_\nu\phi$$

<u>Background</u>: $ds^2 = a^2(\eta)\left[-d\eta^2 + d\bar{x}^2\right]$, $\phi = \phi(\eta)$

$$\Rightarrow \Omega = \sqrt{1 + \lambda\dot{\phi}^2/\phi^4}$$

$$T^\mu{}_\nu = \mathrm{diag}(-\rho, p, p, p) \longrightarrow \begin{cases} T_{00} = a^2\rho \\ T_{ii} = a^2 p \end{cases}$$

$$\Rightarrow \rho = \frac{T_{00}}{a^2} = \frac{1}{a^2}\left\{-a^2\left[\frac{\phi^4}{\lambda}\Omega - V\right] + \Omega^{-1}\phi'^2\right\} =$$

$$= -\left[\frac{\phi^4}{\lambda}\Omega - V\right] + \frac{\Omega^{-1}\phi'^2}{a^2}$$

$$p = \frac{T_{ii}}{a^2} = \frac{1}{a^2}\left\{a^2\left[\frac{\phi^4}{\lambda}\Omega - V\right]\right\} = \frac{\phi^4}{\lambda}\Omega - V$$

$$H^2 = \frac{\rho}{3M_p^2} = \frac{1}{3M_p^2}\left\{-\left[\frac{\phi^4}{\lambda}\Omega - V\right] + \frac{\Omega^{-1}\phi'^2}{a^2}\right\}$$

<u>Notation</u>: $(\ )' \equiv \frac{d}{d\eta}(\ )$, $(\dot{\ }) \equiv \frac{d}{dt}(\ )$

Thus, we got $p = -\rho + \Omega^{-1}\dot{\phi}^2$.

We have inflation ($p \approx -\rho$) provided $\Omega^{-1}\dot{\phi}^2 \ll V, \frac{\phi^4}{\lambda}\Omega$.

The eq. of motion for $\phi(t)$ is:

$$\frac{4\phi^4}{\lambda}\Omega - V'\phi - 2\dot{\phi}^2\Omega^{-1} + \phi\Omega^{-1}[\ddot{\phi} + 3H\dot{\phi}] - \Omega^{-1}\dot{\phi}^2\left[\frac{\lambda}{\Omega\phi^4}\right]\left[2\dot{\phi}^2\Omega^{-1} - \ddot{\phi}\phi\Omega^{-1}\right] = 0$$

During inflation, this eq. can be approximated as follows:

$$\frac{4\phi^4}{\lambda}\Omega - V'\phi + \phi\Omega^{-1}[\ddot{\phi} + 3H\dot{\phi}] \approx 0$$

and the Friedmann eq. reduces to

$$H^2 \approx \frac{1}{3M_p^2}\left(V - \frac{\phi^4}{\lambda}\Omega\right) \longrightarrow \frac{\phi^4}{\lambda}\Omega \approx V - 3M_p^2H^2$$

$$\longrightarrow V - 3M_p^2H^2 - V'\phi + \phi\Omega^{-1}[\ddot{\phi} + 3H\dot{\phi}] \approx 0$$

$$\longrightarrow \left[\frac{V'M_p}{V}\right] \approx \underbrace{\frac{M_p}{\phi}}_{\gtrsim 1} + \frac{M_p\Omega^{-1}[\ddot{\phi} + 3H\dot{\phi}]}{V\phi} - \frac{3M_p^3H^2}{V\phi} \gtrsim 1 \quad (*)$$

$\longrightarrow$ we have inflation but the potential is steep!

Similarly, $\frac{V''M_p^2}{V} \approx \frac{M_p^2}{\phi^2} + \ldots \gtrsim 1$

Perturbations

Let us now derive the eq. for scalar field perturbations $\delta\phi$ in the approximation where we neglect metric perturbations.

If we plug $\phi = \phi_0 + \delta\phi$ into:

$$\frac{4\phi^4}{\lambda}\Omega - \phi V' + \left[2(\partial\phi)^2 + \phi\Box\phi\right]\Omega^{-1} +$$
$$+ \lambda\Omega^{-3}\frac{\partial_\mu\phi}{\phi^4}\left[\partial^\mu\partial^\lambda\phi\,\partial_\lambda\phi\,\phi - 2(\partial\phi)^2\partial^\mu\phi\right] = 0$$

$$\longrightarrow \frac{16\phi_0^3\delta\phi}{\lambda} + \frac{4\phi^4}{\lambda}\delta\Omega + \Big[-2\dot\phi_0\dot{\delta\phi} - \delta\phi(\ddot\phi_0 + 3H\dot\phi_0) +$$
$$+ \phi_0\left(-\ddot{\delta\phi} + a^{-2}\Delta\delta\phi - 3H\dot{\delta\phi}\right) -$$
$$- \left[-2\dot\phi_0^2 - \phi_0(\ddot\phi_0 + 3H\dot\phi_0)\right]\Omega_0^{-2}\delta\Omega -$$
$$+ 3\lambda\,\delta\Omega\,\Omega_0^{-4}\frac{\dot\phi_0}{\phi_0^4}\left[-\phi_0\dot\phi_0\ddot\phi_0 + 2\dot\phi_0^3\right] +$$
$$+ 4\lambda\Omega_0^{-3}\frac{\dot\phi_0\delta\phi}{\phi^5}\left[-\phi_0\dot\phi_0\ddot\phi_0 + 2\dot\phi_0^3\right] -$$
$$- \lambda\Omega_0^{-3}\frac{\dot{\delta\phi}}{\phi^4}\left[-\phi_0\dot\phi_0\ddot\phi_0 + 2\dot\phi_0^3\right] - \lambda\Omega_0^{-3}\frac{\dot\phi_0}{\phi^4}\Big[-\delta\phi\,\dot\phi_0\ddot\phi_0 -$$
$$- \phi_0\dot{\delta\phi}\,\ddot\phi_0 - \phi_0\dot\phi_0\ddot{\delta\phi} + 6\dot\phi_0^2\dot{\delta\phi}\Big] = 0$$

where $\Omega_0 \equiv \sqrt{1 + \lambda \dot{\phi}_0^2/\phi_0^4}$ and

$$\delta\Omega = \frac{\Omega_0^{-1}}{2}\lambda\left\{2\frac{\dot{\phi}_0\delta\dot{\phi}}{\phi_0^4} - 4\frac{\lambda\dot{\phi}_0^2}{\phi_0^5}\delta\phi\right\}$$

Finally - we have argued that if $\exists$ an inflating soln, then the potential is steep in the absence of cancellations between terms in $(*)$. We can argue that $\exists$ a self-consistent inflating soln as follows:

As the scalar rolls down a steep potential, we may begin in the slow-roll regime but we will enter the speed limit regime where $\frac{\lambda\dot{\phi}^2}{\phi^4} \approx -1$.

Then, $\left(\frac{\lambda\dot{\phi}^2}{\phi^4}\right)^{\cdot} \approx 0 \rightarrow \frac{2\lambda\dot{\phi}\ddot{\phi}}{\phi^4} - \frac{4\lambda\dot{\phi}^2}{\phi^5}\dot{\phi} \approx 0 \rightarrow \ddot{\phi} \approx \frac{2\dot{\phi}^2}{\phi}$

and the (background-level) eom becomes

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{\lambda\dot{\phi}^2}{\phi^4}\gamma^3\left(\ddot{\phi} - 2\frac{\dot{\phi}^2}{\phi}\right) - \frac{4\phi^3}{\lambda} - 2\frac{\dot{\phi}^2}{\phi} + \frac{1}{\gamma}\frac{\partial V}{\partial\phi} = 0 \qquad \gamma = \frac{1}{\sqrt{1+\frac{\lambda\dot{\phi}^2}{\phi^4}}}$$

$$\approx \underline{3\frac{\dot{a}}{a}\dot{\phi} - \frac{4\phi^3}{\lambda} + \frac{1}{\gamma}\frac{\partial V}{\partial\phi} = 0}$$

We know $\ddot{a} \propto -(\rho + 3p) = -\left(\frac{\phi^4}{\lambda}\gamma + V + 3\left(\frac{\phi^4}{\lambda\gamma} - V\right)\right)$

$$\approx +\underbrace{\frac{\phi^4}{\lambda}\gamma}_{<0} + \underbrace{2V}_{>0}$$

so $\exists$ an inflating solution if $V \gg \left|\frac{\phi^4}{\lambda}\gamma\right|$. This will hold if $\frac{H\dot{\phi}}{\phi^3/\lambda} \gg 1$, in which case $\frac{\phi^4}{\lambda} \ll \frac{1}{\gamma}\frac{\partial V}{\partial\phi}\phi \approx \frac{1}{\gamma}V$.

$$\frac{H\dot{\phi}}{\phi^3/\lambda} \approx \frac{\sqrt{V}}{M_p}\frac{\phi^2\sqrt{\lambda}}{\phi^3/\lambda} \approx \frac{m\phi}{M_p}\frac{\sqrt{\lambda}}{\phi} \approx \frac{m\sqrt{\lambda}}{M_p} \quad \text{where } V(\phi) = \tfrac{1}{2}m^2\phi^2.$$

There exists an inflating solution for $\frac{m\sqrt{\lambda}}{M_p} \gg 1 \rightarrow$ the opposite of what we would expect for slow-roll!