Problem 1.4

We need to evaluate the wavefunction for the Euclideanised action:

$$S_{Euclidean} = \frac{R_{AdS}^2}{16\pi G_N}[-\int_{\Sigma_4}\sqrt{-g}(R+6) - 2\int_{\partial\Sigma_4} K]$$

where, $K = \frac{1}{2}h^{ab}\partial_n h_{ab}$. The metric is given to be: $ds^2 = d\rho^2 + \sinh^2\rho\ d\Omega_3^2$.
For this metric, $R = -12$. We also find K.

$$\begin{aligned} K &= \frac{1}{2}Tr\Big[\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sinh^2\rho\Omega_3}\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2\sinh\rho\cosh\rho\Omega_3\end{pmatrix}\Big] \\ &= 3\coth\rho \end{aligned}$$

Now the Σ_4 integral gives:

$$\begin{aligned} \int_{\Sigma_4}\sqrt{|g|} &= Vol(S^3)\int_0^{\rho_c} d\rho \sinh^3\rho \\ &= \frac{Vol(S^3)}{8}\int_0^{\rho_c} d\rho(e^\rho - e^{-\rho})^3 \\ &= \frac{-Vol(S^3)}{12} + \lim_{\rho_c\to+\infty}\frac{Vol(S^3)}{8}(\frac{e^{3\rho_c}}{3} - 3e^{\rho_c}) \end{aligned}$$

The $\partial\Sigma_4$ integral gives:

$$\int_{\partial\Sigma_4} K = \lim_{\rho_c\to+\infty} Vol(S^3)\coth\rho_c$$

(since the boundary is at $\rho = +\infty$) $\therefore$ We can now write down the on-shell action as:

$$S_{Euclidean} = \frac{R_{AdS}^2}{16\pi G_N}\Big[\frac{-Vol(S^3)}{2} + \lim_{\rho_c\to+\infty}\frac{3Vol(S^3)}{4}(\frac{e^{3\rho_c}}{3} - 3e^{\rho_c}) - 6\lim_{\rho_c\to+\infty} Vol(S^3)\coth\rho_c\Big]$$

Thus the wave-function:

$$\begin{aligned} \psi &= e^{-S_{Euclidean}} \\ &= \exp\big(\frac{R_{AdS}^2}{32\pi G_N}Vol(S^3)\big) \end{aligned}$$

Problem 2.3

We compute the equal time 2-point function of a massless scalar in a fixed de Sitter background,

$$\Big\langle BD|\phi(\vec{x},\eta)\phi(\vec{x'},\eta)|BD\Big\rangle \ . \tag{1}$$

Fourier transforming (1) we have,

$$\begin{aligned} &\Big\langle BD|\phi(\vec{x},\eta)\phi(\vec{x'},\eta)|BD\Big\rangle \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}e^{i\vec{k'}\cdot\vec{x'}}\Big\langle BD|\phi(\vec{k},\eta)\phi(\vec{k'},\eta)|BD\Big\rangle \ . \end{aligned} \tag{2}$$

Here we can write

$$\phi(\vec{k},\eta) = f_k(\eta)a^\dagger + f_k^*(\eta)a \quad \text{where} \quad f_k(\eta) = \frac{1}{k^{3/2}}(1-ik\eta)e^{ik\eta}\,. \tag{3}$$

The Bunch-Davis vacuum is annihilated by a,

$$[a,a^\dagger] = 1, \quad a|BD\rangle = 0, \quad \langle BD|BD\rangle = 1\,. \tag{4}$$

Using (4) we have the 2-point function in momentum space as

$$\begin{aligned}\Big\langle BD|\phi(\vec{k},\eta)\phi(\vec{k'},\eta)|BD\Big\rangle &= f_k^*(\eta)f_{k'}(\eta) \\ &= \frac{1}{(kk')^{3/2}}(1+ik\eta)(1-ik'\eta)e^{-i(k-k')\eta}\end{aligned} \tag{5}$$

We can also impose the momentum conservation condition, then

$$\begin{aligned}&\Big\langle BD|\phi(\vec{x},\eta)\phi(\vec{x'},\eta)|BD\Big\rangle \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{(1+ik\eta)(1-ik'\eta)}{(kk')^{3/2}}e^{-i(k-k')\eta}e^{i\vec{k}\cdot\vec{x}+i\vec{k'}\cdot\vec{x'}}(2\pi)^3\delta(\vec{k}+\vec{k'}) \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{(1+k^2\eta^2)}{k^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x'})} \\ &= \frac{1}{2\pi^2\Delta x}\int_0^\infty dk\frac{(1+k^2\eta^2)}{k^2}\sin(k\Delta x)\,,\ \Delta x = |\vec{x}-\vec{x'}|,\ k=|\vec{k}| \qquad (6) \\ &= \frac{1}{2\pi^2\Delta x}\left\{\Delta x(1-\gamma) - \lim_{\epsilon\to 0}\ln(\Delta x\epsilon) + \frac{\eta^2}{\Delta x}\right\}. \qquad (7)\end{aligned}$$

Here γ is the Euler - Mascheroni constant. So the finite part of the 2-point function is,

$$\Big\langle BD|\phi(\vec{x},\eta)\phi(\vec{x'},\eta)|BD\Big\rangle = \frac{1}{2\pi^2}\left\{1-\gamma+\frac{\eta^2}{(\Delta x)^2}\right\}. \tag{8}$$

The physical origin of the IR divergence is quite subtle. The correct argument why there is an IR divergence comes from the fact that we sum over all states inside and outside the cosmological horizon. Let us denote $P = \frac{p}{e^{Ht}}$ as the physical momentum. The cosmological horizon is at $P = H$, inside the horizon $P > H$, the number of degrees of freedom is constant and there is no problem. However, outside the horizon for $P < H$, number of degrees of freedom grows with time which is the reason of IR divergence. Therefore, we should write $< \phi(x)\phi(x') > \approx \int_H^\Lambda \frac{dP}{P}$ where H is now our IR cutoff while Λ is the UV cutoff.

Problem 3.4

The rate equation is:-

$$\Delta P_a(n) = -\Sigma_b\gamma_{ba}P_a(n) + \Sigma_b\gamma_{ab}P_b(n)$$

where, $\{a\}$ are the available vacuua, and γ_{ba} is the decay rate per unit volume from vacuua 'a' to vacuua 'b'.
To make the transfer matrix symmetric we rescale the rate variables in the following fashion:

$$\gamma_{ab} = M_{ab} \exp(S_a)$$

where, S_a is the entropy of vacuua 'a', and M is a symmetric matrix. Thus now, our transfer matrix looks like:

$$T = \begin{bmatrix} (-M_{21}e^{S_2} - M_{31}e^{S_3}...) & M_{12}e^{S_1} & .... \\ M_{21}e^{S_2} & (-M_{12}e^{S_1} - M_{32}e^{S_3}...) & ... \\ ... & ... & .... \end{bmatrix}$$

or,

$$T = \begin{bmatrix} (-M_{21}e^{(S_2-S_1)} - M_{31}e^{(S_3-S_1)}...) & M_{12} & .... \\ M_{21} & (-M_{12}e^{(S_1-S_2)} - M_{32}e^{(S_3-S_2)}...) & ... \\ ... & ... & .... \end{bmatrix} \times \exp(S_1 + S_2 + S_3 + ...)$$

Thus we have a symmetric matrix. Now looking at the matrix:

$$T = \begin{bmatrix} (-M_{21}e^{S_2} - M_{31}e^{S_3}...) & M_{12}e^{S_1} & .... \\ M_{21}e^{S_2} & (-M_{12}e^{S_1} - M_{32}e^{S_3}...) & ... \\ ... & ... & .... \end{bmatrix}$$

it is easy to see that we can decrease the rank by one by adding up the rows. Hence the determinant of T is zero. This implies we have a zero eigenvalue. Now the characteristic polynomial of an n $\times$ n square matrix is $\sum_{j=0}^{n}(-1)^j a_j \lambda^j$ where $a_n = 1$, $a_{n-1} =$ Tr A, $a_0 =$ Det A and general a_j is the sum over j-rowed diagonal minors. These minors are all $(-1)^j * C_j$ where C_j is some positive number (and $C_0 = 0$). Therefore, the characteristic polynomial equation is $\sum_{j=0}^{n} C_j \lambda^j = 0$. Because all C_j are positive, therefore λ must be negative such that we get zero on RHS.
Now we want to find the zero eigenvector of the matrix:

$$T = \begin{bmatrix} (-M_{21}e^{S_2} - M_{31}e^{S_3}...) & M_{12}e^{S_1} & .... \\ M_{21}e^{S_2} & (-M_{12}e^{S_1} - M_{32}e^{S_3}...) & ... \\ ... & ... & .... \end{bmatrix}$$

We can write:

$$(T)_{\alpha\beta} = \sum_{\gamma} (-M_{\gamma\alpha} e^{(S_\gamma - S_\alpha)})\delta_{\alpha\beta} + M_{\alpha\beta}$$

We need to satisfy:

$$\begin{aligned}
&\textstyle\sum_\beta && (T_1)_{\alpha\beta}P_\beta = 0 \\
or, \quad &\textstyle\sum_{\beta,\gamma} && (-M_{\gamma\alpha}e^{(S_\gamma - S_\alpha)})\delta_{\alpha\beta}P_\beta + \sum_\beta M_{\alpha\beta}P_\beta = 0 \\
or, \quad &\textstyle\sum_\gamma && (-M_{\gamma\alpha}e^{(S_\gamma - S_\alpha)})P_\alpha + \sum_\beta M_{\alpha\beta}P_\beta = 0 \\
or, \quad &\textstyle\sum_\beta && [-M_{\alpha\beta}(e^{(S_\beta - S_\alpha)}P_\alpha - P_\beta)] = 0 \\
or, \frac{P_\beta}{P_\alpha} &= && \frac{\exp(S_\beta)}{\exp(S_\alpha)}
\end{aligned} \tag{9}$$

Thus the zero eigenvector is:

$$P_a = \exp(S_a) \tag{10}$$

Problem 4.1

a) We consider a model of inflation based on a potential $V(\phi) = \mu^{4-p}\lambda\phi^p$ in the regime $\phi > M_P$ with $0 < p \leq 2$. First note the following equations:

$$H^2 = 8\pi G\rho \quad \rho = \frac{\dot{\phi}^2}{2} + V(\phi) \approx V(\phi) \ . \tag{11}$$

From these equations we can approximate H^2 as follows:

$$H^2 \approx 8\pi G V(\phi) = \frac{1}{M_P^2}\mu^{4-p}\lambda\phi^p \equiv \frac{1}{M_P^2}\kappa\phi^p \tag{12}$$

Also from the equation of motion with the slow-roll condition ($\ddot{\phi} \approx 0$) we can approximate $\dot{\phi}$:

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial\phi} = -p\kappa\phi^{p-1} \quad \Rightarrow \quad \dot{\phi} \approx -\frac{p\kappa}{3H}\phi^{p-1} \tag{13}$$

From these equations we have the ratio,

$$\frac{H}{\dot{\phi}} = -\frac{3}{pM_P^2}\phi \tag{14}$$

Now let us consider the N_e e-folding during inflation:

$$\begin{aligned}
N_e = \int\frac{da}{a} = \int H dt = \int\frac{H}{\dot{\phi}}d\phi = \int_{\phi_i}^{\phi_f}\left[-\frac{3}{pM_P^2}\phi\right]d\phi \\
= -\frac{3}{2pM_P^2}[\phi_f^2 - \phi_i^2] \approx -\frac{1}{pM_P^2}\phi\Delta\phi \ .
\end{aligned} \tag{15}$$

Now in $\phi > M_P$ (so that $M_P < \phi < \phi + \Delta\phi$) we can determine $\Delta\phi$:

$$N_e = -\frac{1}{pM_P^2}\phi\Delta\phi \geq -\frac{1}{pM_P^2}M_P\Delta\phi \quad \Rightarrow \quad \Delta\phi \leq -N_e p M_P = -60pM_P \tag{16}$$

Now from the scalar power spectrum

$$< \xi\xi > \sim (\frac{H^2}{\dot{\phi}})^2 \sim 10^{-10} \tag{17}$$

we can determine the parameter μ:

$$10^{-5} \sim \frac{H^2}{\dot{\phi}} = -\frac{3H^3}{p\kappa}\phi^{1-p} = -\frac{3\sqrt{\kappa}}{p}\frac{\phi^{p/2+1}}{M_P^3} \geq \frac{3\sqrt{\kappa}}{pM_P^{2-p/2}} \tag{18}$$

$$\Rightarrow \quad \kappa = \mu^{4-p}\lambda \leq 10^{-11}pM_P^{4-p} \tag{19}$$

b) Given the action

$$S = \int d^4x\sqrt{-g}\Big[(\partial\phi)^2 + \mu^{4-p}\lambda\phi^p\Big] \tag{20}$$

The quantum corrections are of order $(\frac{\kappa}{M_{Pl}^{(4-p)}})^{power}$. Therefore, the corrections are small as we expect. Now, the slow-roll conditions $M_{Pl}^2\frac{V''}{V}\ \frac{M_{Pl}^2}{\phi^2}$. To check that this is small we have to consider $\phi >> M_{Pl}$ and to show that quantum corrections are small. However, the contributions to the potentially dangerous operators like e.g. $\frac{\phi^6}{M_{Pl}}$ that are huge for $\phi >> M_{Pl}$ must have a factor $(\frac{\kappa}{M_{Pl}^{(4-p)}})^{power}$ and since this is very small as $\kappa \to 0$, the system is stable.

c) In the UV completion we would have to solve the eta problem, ie. the higher order terms in the Lagrangian that are suppressed by $\frac{1}{M_{Pl}}$ can not be ignored because they can still give the $O(1)$ contribution. E.g. the gravitational wave problem, where we need to sum the infinite series of corrections where all terms are important because $\Delta\phi > M_{Pl}$. Thus one wants to find some additional symmetry that protects us from writing higher order terms.

Problme 5.4

Parameter dependence of the cosmological model

The equations we have are, the Friedmann equation

$$H^2 = \frac{8\pi Ga^2}{3}\rho_{tot} \ , \tag{21}$$

and the energy-momentum conservation

$$\phi = -\frac{4\pi Ga^2}{k^2}\left(\rho + 3\frac{\dot{a}}{a}\frac{f}{k}\right) \ , \ \dot{\phi} = -\frac{\dot{a}}{a}\phi + \frac{4\pi Ga^2 f}{k} \tag{22}$$

Here the equations are in conformal time. And the equations for perturbations in fluids (assuming the tight coupling for simplicity) are

$$\dot{\delta}_b = -kv_b + 3\dot{\phi}\ , \quad \dot{v}_b = -\frac{\dot{a}}{a}v_b + k\phi \tag{23}$$

$$\dot{\delta}_c = -kv_c + 3\dot{\phi}\ , \quad \dot{v}_c = -\frac{\dot{a}}{a}v_c + k\phi \tag{24}$$

Here the tight coupling condition is

$$\delta_b = \frac{3}{4}\delta_\gamma\ ,\ v_b = v_\gamma \tag{25}$$

Now, we want to know which parameter are involved to evolve these equations. First we need initial conditions and the initial conditions depend on A and n_s from inflation. To get $H = \frac{\dot{a}}{a}$ in the above equations, we use $\rho_b + \rho_c + \rho_\nu + \rho_\gamma$ which shows the dependence on $\Omega_b h^2, \Omega_c h^2$, and h. We also see that these parameters later enter the equations for perturbations $(\dot{\delta}_i, \dot{v}_i)$. Since Ω_Λ and Ω_b are much smaller than Ω_c and before recombination, we are mostly in radiation-dominated era, Ω_Λ and Ω_b can be ignored approximately.

Effect of changing distance to the last scattering surface

The effective oscillatory equation we get for linear perturbations from reheating up to recombination have the form (combining equations for $\dot{\delta}$ and $\dot{v}$):

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k + \left(\frac{k^2}{3} - 4\pi G\rho_{tot}\right)\delta_k = 0\ . \tag{26}$$

This equation has an answer of the form:

$$\delta_k = A\sin(c_s k\tau)\ . \tag{27}$$

This means that different modes oscilate with the speed of sound, c_s up to the last scattering. Changing the distance to the last scattering means that we change the time as which these modes can oscilate. Therefore the extremums of $\sin(c_s k\tau)$ happen at different k (or l). This causes the peaks in the power spectrum to shift to right or left.

$$\sin(c_s k\tau) = 0 \quad \rightarrow\ c_s k\tau_{atLS} = n\pi \quad \rightarrow\ k = \frac{n\pi}{c_s\tau_{atLS}} \tag{28}$$

Degeneracy between the parameters

Different parameters can produce similar effects on perturbations at the last scattering as follows: One example is that h can change the distance to the last scattering surface and therefore shift the peaks to left and right. But Ω_b determines the photon/baryon ratio and therefore the sound speed in plasma so that

$$c_s k\tau_{atLS} = n\pi \tag{29}$$

is dependent on both c_s and τ_{atLS}. Since $\Omega_m, \Omega_b, \Omega_c$, it is also degenerated with h. Another example is that changing Ω_Λ but keeping $\Omega_\Lambda + \Omega_c$ fixed together with Ω_b fixed would change h, but we can fix $\Omega_\Lambda + \Omega_b$ and change Ω_c to get the same effect.

Problem 6.1

Given;

$$S = \int \sqrt{-g} d^4x \{F'(A)(R - A) + F(A)\}$$

Varying auxiliary field A :-

$$\frac{\delta S}{\delta A} = \int \sqrt{-g} d^4x \{F''(A)(R - A) - F'(A) + F'(A)\}$$

Thus we obtain: $R = A$. Plugging this into the action we have:

$$S = \int \sqrt{-g} d^4x F(R)$$

Now we do a conformal transformation:

$$g_{\mu\nu} \to g_{\mu\nu} \exp(\sigma)$$

Therefore, $\sqrt{-g} \to \sqrt{-g} \exp(2\sigma)$ and $R \to \exp(-\sigma)\{R - \frac{3}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma +$ total derivative $\}$
Demanding conformal invariance of the action gives:-

$$S = \int \sqrt{-g} d^4x \exp(2\sigma) \left(F'(A)\{e^{-\sigma}(R - \frac{3}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma) - A\} + F(A) \right)$$

We want to write this as:

$$S = \int \sqrt{-g} d^4x \left(R - \frac{3}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V(\sigma) \right)$$

So we have the following equation,

$$\begin{aligned} e^\sigma F'(A) &= 1 \\ \therefore e^\sigma &= -\log F'(A) \end{aligned}$$

And also $V(\sigma) = e^{2\sigma} F'(A) A - e^{2\sigma} F(A) = \frac{A}{F'(A)} - \frac{F(A)}{(F'(A))^2}$.(proved)

Problem 6.1

We need to find the graviton propagator in D dimensions. We know that the polarization tensor has to be a $(D-2) \times (D-2)$ dimensional traceless transverse symmetric matrix. Which means that there are $\frac{D(D-3)}{2}$ number of degrees of freedom. Now looking at the polarization sum tensor (to which the propagator has to be proportional to) we can see that the following structure works well when the indices μ, ν, α, β run from 1 to $D-2$, that is the transverse directions only.

$$N_{\mu\nu\alpha\beta} = -c\ \eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\alpha\nu}$$

We fix the coefficient c to be equal to $\frac{2}{D-2}$ by demanding the tracelessness condition. By construction the propagator is symmetric. Now we have to make it Lorentz invariant, i.e, we should be able to write it for the indices running over all the spacetime directions. Thus we have to add a non-Lorentz invariant part which should cancel the contributions coming when the indices take values 0 or $D-1$. The tensor we have at our disposal other than $\eta_{\mu\nu}$ is the momentum vector k^μ. Thus we can write it as a bilinear of the form $k^\alpha \bar{k}^\beta$ where we define our vetors in the following fashion:

$$k^\mu = \begin{pmatrix} E \\ 0 \\ . \\ . \\ 0 \\ E \end{pmatrix}$$

and,

$$\bar{k}^\mu = \begin{pmatrix} -E \\ 0 \\ . \\ . \\ 0 \\ E \end{pmatrix}$$

We add these bilinears in the required symmetric fashion. Therefore now with the indices running over all values from 0 to $D-1$ our propagator looks like:

$$\begin{aligned} N_{\mu\nu\alpha\beta} &= -\frac{2}{D-2}\ \eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\alpha\nu} + \frac{1}{2k\cdot\bar{k}}[a_1\ k_{(\mu}\bar{k}_{\nu)}\eta_{\alpha\beta} + a_2\ k_{(\alpha}\bar{k}_{\beta)}\eta_{\mu\nu} \\ &\quad + a_3\ k_{(\mu}\bar{k}_{\alpha)}\eta_{\nu\beta} + ... + a_6\ k_{(\nu}\bar{k}_{\beta)}\eta_{\mu\alpha}] \end{aligned}$$

The coefficients a_1, a_2...., a_6 are now fixed by demanding cancellation of the extra previously absent piece coming from the first three terms. In particular we need to look at the following

μ	ν	α	β	Needed cancellation
0	0	0	0	$\frac{2}{D-2} - 2$
0	0	i	i	$\frac{-2}{D-2}$
....				
$D-1$	i	i	$D-1$	1

The table thus can be constructed from the extra pieces, and thus coefficients can be obtained consistently. Finally we end up with the following expression for the propagator:

$$\begin{aligned} N_{\mu\nu\alpha\beta} &= -\frac{2}{D-2}\,\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\alpha\nu} + \frac{1}{2k\cdot\bar{k}}[\frac{-2}{D-2}(k_{(\mu}\bar{k}_{\nu)}\eta_{\alpha\beta} + k_{(\alpha}\bar{k}_{\beta)}\eta_{\mu\nu}) \\ &\quad + k_{(\mu}\bar{k}_{\alpha)}\eta_{\nu\beta} + \ k_{(\mu}\bar{k}_{\beta)}\eta_{\nu\alpha} + \ k_{(\nu}\bar{k}_{\alpha)}\eta_{\mu\beta} + \ k_{(\nu}\bar{k}_{\beta)}\eta_{\mu\alpha}]. \end{aligned}$$