PiTP: Assigned Problems

Group 2

July 22, 2011

Problem 2: Maldacena

From the question 1.1, we have

$$\phi(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \left(f_k(\eta)\hat{a}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} + f_k(\eta)^* \hat{a}^\dagger_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \right)$$

where $k = |\mathbf{k}|$ and

$$f_k(\eta) = \frac{1}{\sqrt{2k^3}}(1 + ik\eta)e^{-ik\eta}.$$

To deal with the interaction perturbatively, we make use of interaction picture where the background evolution according to the free theory is described by Heisenberg picture and the perturbative corrections are expressed in Schrodinger picture. The ϕ^3 interaction leads to the Hamiltonian

$$H_{\mathrm{I}}(\eta) = \frac{g}{6\eta^4} \int dw^3 \phi(\eta, \mathbf{w})^3.$$

Let us define the perturbative evolution operator

$$U(\eta, \eta_0) = \exp\left(-i \int_{\eta_0}^{\eta} d\eta' H_{\mathrm{I}}(\eta') \right).$$

In computing three-point function, we assume that the quantum state in the far past was Bunch-Davies vacuum. Then the equal-time correlator is given by

$$\langle \mathrm{BD}| U(-\infty, \eta)^\dagger \phi(\eta, \mathbf{x})\phi(\eta, \mathbf{y})\phi(\eta, \mathbf{z}) U(\eta, -\infty)|\mathrm{BD}\rangle$$

Expanding U's to the leading order in the coupling g, the amplitude becomes

$$\langle \phi(\eta)^3 \rangle = \frac{ig}{6} \int d^3w \int_{-\infty}^{\eta} d\eta' \frac{1}{\eta'^4} \langle \mathrm{BD}| \left[\phi(\eta', \mathbf{w})^3, \phi(\eta, \mathbf{x})\phi(\eta, \mathbf{y})\phi(\eta, \mathbf{z}) \right] |\mathrm{BD}\rangle.$$

We proceed to normal order the expression by splitting ϕ into positive and negative frequency part as

$$\phi(\eta, \mathbf{x}) = \phi^+(\eta, \mathbf{x}) + \phi^-(\eta, \mathbf{x})$$

where

$$\phi^+(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} f_k(\eta) a_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} = \phi^-(\eta, \mathbf{x})^\dagger.$$

Note that

$$\phi^+|\mathrm{BD}\rangle = \langle\mathrm{BD}|\phi^- = 0$$

and

$$\begin{aligned} [\phi^+(\eta, \mathbf{x}), \phi^-(\eta', \mathbf{y})] &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^3}(1 + ik\eta)(1 - ik\eta')e^{-ik(\eta-\eta')-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\ &\equiv D(x, y). \end{aligned}$$

We have

$$\begin{aligned} &\langle\mathrm{BD}| \left[\phi(\eta', \mathbf{w})^3, \phi(\eta, \mathbf{x})\phi(\eta, \mathbf{y})\phi(\eta, \mathbf{z})\right] |\mathrm{BD}\rangle = 6D(x, w)D(y, w)D(z, w) \\ &+3D(w, w)\left[D(x, y)D(z, w) + D(y, z)D(x, w) + D(z, x)D(y, w)\right] - \text{c.c.}. \end{aligned}$$

Now we have to handle integrals. The spatial integral with respect to $\mathbf{w}$ results in delta function $\delta(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3})$. The time integration is divergent. For example, the first term gives rise to

$$\int_{-\infty}^{\eta} d\eta' \frac{1}{\eta'^4}(1 + ik_1\eta')(1 + ik_2\eta')(1 + ik_3\eta')e^{-i(k_1+k_2+k_3)(\eta'-\eta)}.$$

As $\eta \to -\infty$, the exponential factor oscillates rapidly. In this ϕ^3 theory, however, because of the factor η'^{-4} coming from the determinant of the metric, the integral is convergent and the oscillatory behaviour doesn't become a problem. This integration doesn't result in a simple function of η. If you were dealing with four-point function in ϕ^4 theory, we would get a term without surpression by power of η' and therefore the integral would be divergent. In such cases, to kill off the divergence, we need to introduce a tilt of the contour $\eta \to -\infty(1 + i\epsilon)$ with an infinitesimal parameter ϵ. This prescription works for the following three terms. For the complex conjugate terms, the contour should naturally be $\eta \to -\infty(1 - i\epsilon)$, which constitute the Keldysh contour described in the lecture.

Problem 8: Creminelli

If one drops the assumption that $\Gamma << H_{end}$, one has to consider the decay explicitely. The equations describing the decay of the matter part into radiation read

$$\dot{\varrho}_m + (3H + \Gamma)\,\varrho_m = 0$$

$$\dot{\varrho}_r + 4H\,\varrho_r - \Gamma\varrho_m = 0$$

$$H^2 = \frac{8\pi G}{3}\left(\varrho_m + \varrho_r\right)$$

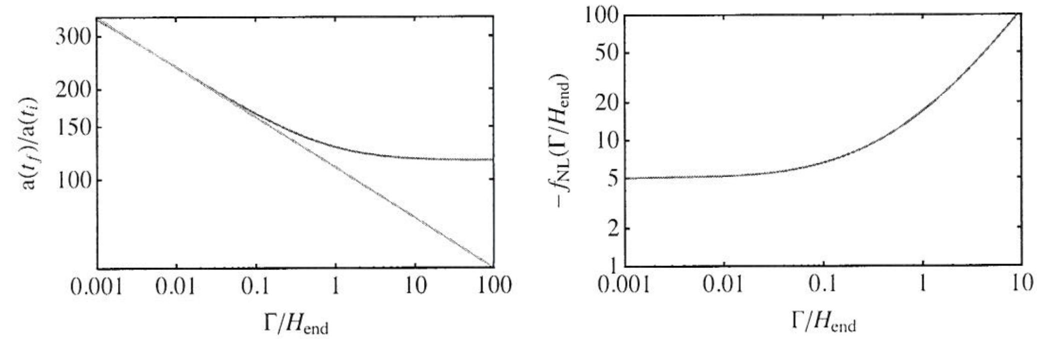

Figure 1: Left: Dependence of the growth of the scale factor during reheating on the value of Γ/H_{end}. Right: Derived dependence of the local non Gaussianity parameter f_{NL}^{local} on Γ/H_{end}.

Measuring time in units of H_{end}, i.e. the Hubble rate at the end of inflation

$$\tau = H_{end}t$$

one gets

$$\dot{\Omega}_m(\tau) + \left(3\tilde{H} + \frac{\Gamma}{H_{end}}\right)\Omega_m(\tau) = 0$$

$$\dot{\Omega}_r(\tau) + 4\tilde{H}\Omega_r(\tau) - \frac{\Gamma}{H_{end}}\Omega_m(\tau) = 0$$

$$\tilde{H}^2 = (\Omega_m(\tau) + \Omega_r(\tau))$$

where $\tilde{H}$ is just $\tilde{H} = \frac{da}{d\tau}/a$

Solving these results numerically gives the dependence of the growth during reheating shown in Fig. 1 on the left. This is related to ζ by

$$\zeta(\Gamma) = \ln\left(\frac{a(t_f)}{a(t_i)}\right)$$

By using the definition of f_{NL}^{local}

$$\zeta = \zeta_g - \frac{3}{5}f_{NL}^{local}\zeta_g^2 + \ldots$$

one finds that it in this case it may be determined by

$$f_{NL}^{local} = -\frac{5}{3}\frac{1}{2}\frac{\frac{\partial^2\zeta}{\partial\Gamma^2}}{\left(\frac{\partial\zeta}{\partial\Gamma}\right)^2}$$

For $\Gamma << H_{end}$ this starts with the value of $f_{NL} = -5$ derived in the lecture. For higher Γ the non Gaussian contribution increases.

Problem 2: Susskind

Let us start with an n - sphere, embedded in an $n+1$ - dimensional Euclidean space,

$$(X^0)^2 + (X^1)^2 + ... + (X^n)^2 = 0, \tag{1}$$

and foliate it by $n-2$ - dimensional spheres of varying radius r,

$$\begin{aligned} X^0 &= \sqrt{R^2 - r^2}\sin\theta, \\ X^1 &= \sqrt{R^2 - r^2}\cos\theta, \\ X^2 &= r\cos\theta_1, \\ X^3 &= r\cos\theta_1\cos\theta_2 \\ X^3 &= r\cos\theta_1\sin\theta_2 \\ &\dots \end{aligned} \tag{2}$$

The space-like slices of constant r and θ represent $n-2$ - dimensional spheres of radius r, as intended. This parametrization is useful, because the angle θ is a "polar" angle in the $X^0 - X^1$ plane, and therefore the metric does not depend on it - eventually it will represent the time variable once we analytically continue to Minkowski space. The metric on the n - sphere, in terms of the new variables can be written as follows,

$$\begin{aligned} ds^2 = (dX^0)^2 + (dX^1)^2 + (dX^2)^2 + ... &= \frac{r^2dr^2}{R^2 - r^2} + (R^2 - r^2)d\theta^2 + dr^2 + r^2 d\Omega^2_{n-2} \\ &= R^2\left[\left(1 - \frac{r^2}{R^2}\right)d\theta^2 + \frac{dr^2}{R^2 - r^2}\right] + r^2 d\Omega^2_{n-2}. \end{aligned}$$

One can now do an analytic continuation into de Sitter - a hyperboloid, embedded in Minkowski spacetime:

$$X^0 \to -iX^0, \quad \theta \to \frac{it}{R}. \tag{3}$$

This brings the metric into the final, static dS form,

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)dt^2 + \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2 d\Omega^2_{n-2}. \tag{4}$$

Problem 3: Silverstein

Given the action

$$S = -\int d^4x\sqrt{-g}\left[\frac{\phi^4}{\lambda}\sqrt{1 - \frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}} - V(\phi)\right]$$

we seek to show that inflation can occur in this theory for a potential which does not satisfy the slow roll conditions. We first derive the equation of motion for the field.

$$\delta S = -\int d^4x\sqrt{-g}\left[4\frac{\phi^3}{\lambda}\delta\phi\sqrt{1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}} - \frac{1}{2}\phi^4\left(\frac{2\frac{\partial_\mu\phi\,\partial^\mu\delta\phi}{\phi^4} - 4\frac{\partial_\mu\phi\,\partial^\mu\phi}{\phi^5}\delta\phi}{\sqrt{1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}}}\right) - \frac{dV}{d\phi}\delta\phi\right]$$

$$= -\int d^4x\sqrt{-g}\left[4\frac{\phi^3}{\lambda}\sqrt{1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}} + \nabla^\mu\left(\frac{\partial_\mu\phi}{\sqrt{1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}}}\right) + 2\frac{\frac{\partial_\mu\phi\,\partial^\mu\phi}{\phi}}{\sqrt{1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}}} - \frac{dV}{d\phi}\right]\delta\phi$$

$$\frac{\delta S}{\delta\phi} = -\sqrt{-g}\left[4\frac{\phi^3}{\lambda}\sqrt{1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}} + \frac{\Box\phi + 2\frac{\partial_\mu\phi\,\partial^\mu\phi}{\phi}}{\sqrt{1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}}} + \frac{\lambda}{\phi^4}\frac{\left(\partial_\mu\phi\partial_\alpha\phi\partial^\alpha\partial^\mu\phi - 2\frac{(\partial_\alpha\phi\partial^\alpha\phi)^2}{\phi}\right)}{\left(1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}\right)^{3/2}} - \frac{dV}{d\phi}\right]$$

$$0 = 4\frac{\phi^3}{\lambda}\sqrt{1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}} + \frac{\Box\phi + 2\frac{\partial_\mu\phi\,\partial^\mu\phi}{\phi}}{\sqrt{1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}}} + \frac{\lambda}{\phi^4}\frac{\left(\partial_\mu\phi\partial_\alpha\phi\partial^\alpha\partial^\mu\phi - 2\frac{(\partial_\alpha\phi\partial^\alpha\phi)^2}{\phi}\right)}{\left(1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}\right)^{3/2}} - \frac{dV}{d\phi}$$

$$0 = \frac{4\phi^4/\lambda - 6\partial_\mu\phi\,\partial^\mu\phi}{\phi\left(1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}\right)^{3/2}} + \frac{\Box\phi}{\sqrt{1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}}} + \frac{\lambda}{\phi^4}\frac{\partial_\mu\phi\partial_\nu\phi\partial^\mu\partial^\nu\phi}{\left(1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}\right)^{3/2}} - \frac{dV}{d\phi}$$

For a homogeneous field $\phi(t)$ in de Sitter space with metric $ds^2 = -dt^2 + e^{2Ht}dx^2$,

$$\boxed{\frac{dV}{d\phi} = \frac{4\phi^4/\lambda + 6\dot\phi^2}{\phi\left(1+\frac{\lambda\dot\phi^2}{\phi^4}\right)^{3/2}} - \frac{\ddot\phi + 3H\dot\phi}{\sqrt{1+\frac{\lambda\dot\phi^2}{\phi^4}}} + \frac{\lambda}{\phi^4}\frac{\dot\phi^2\ddot\phi}{\left(1+\frac{\lambda\dot\phi^2}{\phi^4}\right)^{3/2}}}$$

We must now derive the condition to have inflation, *i.e.* $w \approx -1$: the stress-energy tensor for ϕ is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}}$$

$$= -\frac{\partial_\mu\phi\partial_\nu\phi}{\sqrt{1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}}} - g_{\mu\nu}\left(\frac{\phi^4}{\lambda}\sqrt{1-\frac{\lambda(\partial_\mu\phi\,\partial^\mu\phi)}{\phi^4}} - V(\phi)\right)$$

The energy density is T_{00} and the pressure is the coefficient of $g_{\mu\nu}$ in the stress-energy tensor:

$$\rho = \frac{\phi^4/\lambda}{\sqrt{1+\frac{\lambda\dot\phi^2}{\phi^4}}} - V(\phi)\,, \qquad p = -\frac{\phi^4}{\lambda}\sqrt{1+\frac{\lambda\dot\phi^2}{\phi^4}} + V(\phi)$$

In the limit where $\lambda\dot\phi^2/\phi^4$ is negligible

$$\rho \approx -\frac{1}{2}\dot\phi^2 + \frac{\phi^4}{\lambda} - V(\phi)\,, \qquad p \approx -\frac{1}{2}\dot\phi^2 - \frac{\phi^4}{\lambda} + V(\phi)$$

and hence the condition that $w \approx -1$ is satisfied when

$$\boxed{\dot{\phi}^2 \ll \frac{\phi^4}{\lambda} - V(\phi)}$$

In the same limit of small $\lambda\dot{\phi}^2/\phi^4$, also neglecting $\ddot{\phi}$, the equation of motion is given by

$$\frac{dV}{d\phi} = \frac{4\phi^3}{\lambda} - 3H\dot{\phi}$$

In the case of arbitrarily small λ, we will find that a slow-roll parameter ϵ of $\mathcal{O}(1)$ is nevertheless consistent with an inflating background. The Friedmann equation in this limit gives

$$H^2 \sim \frac{1}{M_{\rm Pl}^2}\left(\frac{\phi^4}{\lambda} - V\right) \Rightarrow \frac{V}{M_{\rm Pl}} \sim \frac{\phi^4}{M_{\rm Pl}\lambda}$$

and from the equation of motion, $V'(\phi) \sim \phi^3/\lambda$ and thus the slow roll parameter $\epsilon = (M_{\rm Pl}V'/V)^2$ scales as

$$\boxed{\epsilon \sim \left(\frac{M_{\rm Pl}}{\phi}\right)^2}$$

In the same limit, we consider a small perturbation around the homogeneous solution $\phi(t,\vec{x}) = \phi_0(t) + \delta\phi(t,\vec{x})$. To linear order in the perturbation,

$$V'(\phi) + \delta\phi V''(\phi) = \frac{4\phi^3}{\lambda} + \frac{12\phi^2\delta\phi}{\lambda}$$

and because $\phi(t)$ is a background solution,

$$\boxed{V''(\phi) = \frac{12\phi^2}{\lambda}}$$

Thus the second slow-roll parameter $\eta = M_{\rm Pl}V''/V$ is of order

$$\boxed{\eta \sim \frac{M_{\rm Pl}}{\phi^2}}$$

The value of ϕ itself is unconstrained by the assumptions we have made, and for $\phi^2 \sim M_{\rm Pl}$, the slow-roll parameters are

$$\epsilon \sim M_{\rm Pl}\,, \qquad \eta \sim 1$$

Thus the slow-roll condition is clearly not met, and yet inflation occurs.

1 Maldacena 2

From the question 1.1, we have

$$\phi(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \left(f_k(\eta)\hat{a}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} + f_k(\eta)^* \hat{a}^\dagger_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \right)$$

where $k = |\mathbf{k}|$ and

$$f_k(\eta) = \frac{1}{\sqrt{2k^3}}(1 + ik\eta)e^{-ik\eta}.$$

To deal with the interaction perturbatively, we make use of interaction picture where the background evolution according to the free theory is described by Heisenberg picture and the perturbative corrections are expressed in Schorodinger picture. The ϕ^3 interaction leads to the Hamiltonian

$$H_{\mathrm{I}}(\eta) = \frac{g}{6\eta^4} \int dw^3 \phi(\eta, \mathbf{w})^3.$$

Let us define the perturbative evolution operator

$$U(\eta, \eta_0) = \exp\left(-i \int_{\eta_0}^{\eta} d\eta' H_{\mathrm{I}}(\eta') \right).$$

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$$\langle \mathrm{BD}| U(-\infty, \eta)^\dagger \phi(\eta, \mathbf{x})\phi(\eta, \mathbf{y})\phi(\eta, \mathbf{z}) U(\eta, -\infty) |\mathrm{BD}\rangle$$

Expanding U's to the leading order in the coupling g, the amplitude becomes

$$\langle \phi(\eta)^3 \rangle = \frac{ig}{6} \int d^3w \int_{-\infty}^{\eta} d\eta' \frac{1}{\eta'^4} \langle \mathrm{BD}| \left[\phi(\eta', \mathbf{w})^3, \phi(\eta, \mathbf{x})\phi(\eta, \mathbf{y})\phi(\eta, \mathbf{z}) \right] |\mathrm{BD}\rangle.$$

We proceed to normal order the expression by splitting ϕ into positive and negative frequency part as

$$\phi(\eta, \mathbf{x}) = \phi^+(\eta, \mathbf{x}) + \phi^-(\eta, \mathbf{x})$$

where

$$\phi^+(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} f_k(\eta) a_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} = \phi^-(\eta, \mathbf{x})^\dagger.$$

Note that

$$\phi^+|\mathrm{BD}\rangle = \langle \mathrm{BD}|\phi^- = 0$$

and

$$\begin{aligned} [\phi^+(\eta, \mathbf{x}), \phi^-(\eta', \mathbf{y})] &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^3} (1 + ik\eta)(1 - ik\eta') e^{-ik(\eta - \eta') - i\mathbf{k}\cdot(\mathbf{x} - \mathbf{y})} \\ &\equiv D(x, y). \end{aligned}$$

We have

$$\langle \mathrm{BD}| \left[\phi(\eta',\mathbf{w})^3, \phi(\eta,\mathbf{x})\phi(\eta,\mathbf{y})\phi(\eta,\mathbf{z})\right] |\mathrm{BD}\rangle = 6D(x,w)D(y,w)D(z,w)$$
$$+3D(w,w)\left[D(x,y)D(z,w) + D(y,z)D(x,w) + D(z,x)D(y,w)\right] - \text{c.c.}.$$

Now we have to handle integrals. The spatial integral with respect to $\mathbf{w}$ results in delta function $\delta(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})$. The time integration is divergent. For example, the first term gives rise to

$$\int_{-\infty}^{\eta} d\eta' \frac{1}{\eta'^4}(1+ik_1\eta')(1+ik_2\eta')(1+ik_3\eta')e^{-i(k_1+k_2+k_3)(\eta'-\eta)}.$$

As $\eta \to -\infty$, the exponential factor oscillates rapidly. In this ϕ^3 theory, however, because of the factor η'^{-4} coming from the determinant of the metric, the integral is convergent and the oscillatory behaviour doesn't become a problem. This integration doesn't result in a simple function of η. If you were dealing with four-point function in ϕ^4 theory, we would get a term without surpression by power of η' and therefore the integral would be divergent. In such cases, to kill off the divergence, we need to introduce a tilt of the contour $\eta \to -\infty(1+i\epsilon)$ with an infinitesimal parameter ϵ. This prescription works for the following three terms. For the complex conjugate terms, the contour should naturally be $\eta \to -\infty(1-i\epsilon)$, which constitute the Keldysh contour described in the lecture.

2 Zaldarriaga 2

We are only concerned with scalar perturbations. We take the uniform density slicing. The remaining spatial degrees of gauge freedom are used to set the spatial to be proportional to δ_{ij}. After solving the Einstein equations for a dust-filled universe (arXiv:0806.1016), we have the following metric;

$$ds^2 = -dt^2 - \frac{4}{5H}\partial_i\zeta_0 dt dx^i + a^2(1+2\zeta_0)\delta_{ij}dx^i dx^j.$$

Here the curvature perturbation ζ_0 is independent of time.

From the metric, we can construct a Lagrangian

$$L = -\left(\frac{dt}{d\lambda}\right)^2 - \frac{4}{5H}\partial_i\zeta_0\frac{dt}{d\lambda}\frac{dx^i}{d\lambda} + a^2(1+2\zeta_0)\delta_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}.$$

Photon momentum P^μ can be regarded as an affinely parametrised tangent vector along a null-geodesic, therefore we define

$$P^0 = \frac{dt}{d\lambda} \qquad P^i = \frac{dx^i}{d\lambda}.$$

Varying t in the Lagrangian, we obtain a geodesic equation

$$\frac{dp^0}{d\lambda} + \frac{2}{5H}\partial_i\zeta_0\frac{dp^i}{d\lambda} + \frac{2}{5H}\partial_i\partial_j\zeta_0 p^i p^j + a\dot{a}(1+2\zeta_0)\delta_{ij}p^i p^j = 0$$

with the null-condition expressed as $L = 0$.

Since the fluid velocity u^μ coincides with the unit normal of constant time hypersurface for dust, we have

$$u^\mu = (1, 0, 0, 0)$$

and

$$P^\mu u_\mu = P^\mu u^\nu g_{\mu\nu} = -P^0 - \frac{2}{5H}\partial_i\zeta_0 p^i.$$

Using the geodesic equation and null-condition, we derive

$$\frac{d}{d\lambda}(P^\mu u_\mu) = -Hp^0(P^\mu u_\mu) - \frac{p^0}{5}\partial_i\zeta_0 p^i.$$

Note that

$$p^0\frac{da}{dt} = \frac{da}{d\lambda} \qquad p^i\partial_i\zeta_0 = \frac{d}{d\lambda}\zeta_0$$

since the derivatives are taken along the path of the photon. Adding an irrelevant second order term, we obtain

$$\frac{d}{d\lambda}(P^\mu u_\mu) = \frac{d}{d\lambda}\left(-\ln a - \frac{1}{5}\zeta_0\right)(P^\mu u_\mu)$$

which solves exactly to give

$$(P^\mu u_\mu) \propto a^{-1}e^{-\frac{1}{5}\zeta_0}.$$

In the tight couplilng limit, the uniform density slice for dust sees uniform density of photons as well. Therefore the temperature fluctuation is just the gravitational redshift of the photons and we conclude

$$\frac{\delta T}{T} = -\frac{1}{5}\zeta_0$$

That is, the curvature fluctuation at recombination is projected onto the temperature map we obeserve.

The standard Sachs-Wolfe effect in the Newtonian gauge is given by

$$\frac{\delta T}{T} = \frac{1}{4}\delta_\gamma + \Psi$$

where δ_γ is the density perturbation of photons on the last scattering surface and Ψ is the difference of gravitational potential between the last scattering and the observation. According to arXiv:0806.1016, the gauge transformation from ζ-gauge to Newtonian gauge is given by

$$t \to t + \frac{2\zeta_0}{5H}.$$

This means the Newtonian matter density perturbation is given by

$$\delta_m = \frac{6}{5}\zeta_0.$$

Again using the tight coupling approximation, we have

$$\frac{1}{4}\delta_\gamma = \frac{1}{3}\delta_m = \frac{2}{5}\zeta_0.$$

Since the potential is related to ζ_0 by

$$\Psi = -\frac{3}{5}\zeta_0,$$

we recover the formula obtained for ζ-gauge.

Arkani-Hamed 3 (Group 2)

Superluminality in the Cubic Galileon

Static, spherically symmetric solution

Let us consider a theory, defined by the following Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \pi)^2 - \frac{c}{6\Lambda^3}\pi \varepsilon_{\mu\alpha\rho\gamma}\varepsilon_{\nu\beta\rho\gamma}\partial_\mu\partial_\nu\pi\partial_\alpha\partial_\beta\pi + \frac{1}{M_{\rm Pl}}\pi T. \tag{1}$$

The scalar self-interaction is (up to a total derivative) the cubic Galileon in disguise - this form will allow us to perform all computations below trivially (all other Galileons are obvious generalizations of this representation of cubic one and are obtained by simply adding powers of $\partial^2\pi$ and contracting the corresponding indices with the two ε-s. This by the way explains why there are no more than two time derivatives in the equations of motion, why there are exactly $D+1$ of them in D dimensions, etc.). Variation by π yields,

$$\Box\pi - \frac{c}{2\Lambda^3}\varepsilon_{\mu\alpha\rho\gamma}\varepsilon_{\nu\beta\rho\gamma}\partial_\mu\partial_\nu\pi\partial_\alpha\partial_\beta\pi + \frac{1}{M_{\rm Pl}}T = 0, \tag{2}$$

which, upon expanding the product of two ε-s, yields,

$$\Box\pi - \frac{c}{\Lambda^3}\left((\partial_\mu\partial_\nu\pi)^2 - (\Box\pi)^2\right) = -\frac{1}{M_{\rm Pl}}T. \tag{3}$$

The quantity on the lhs can be written as a total derivative (another thing one can trivially see from the ε representation),

$$\partial_\mu\left(\partial_\mu\pi - \frac{c}{\Lambda^3}\left(\partial_\nu\pi\partial_\mu\partial_\nu\pi - \partial_\mu\pi\Box\pi\right)\right) = -\frac{1}{M_{\rm Pl}}T. \tag{4}$$

For spherically symmetric, static solutions in the presence of a point source, $T = -M\delta^3(\mathbf{x})$, the latter equation reduces to,

$$\frac{1}{r^2}\partial_r\left(r^2\partial_r\pi + \frac{2c}{\Lambda^3}r(\partial_r\pi)^2\right) = \frac{M}{M_{\rm Pl}}\delta^3(\mathbf{x}), \tag{5}$$

Multiplying both sides, integrating, denoting r-derivatives by primes and ignoring positive factors of order one, one arrives at the following quadratic equation for π'

$$\pi'^2 + \frac{\Lambda^3}{2c}r\pi' - \frac{\Lambda^3}{2cr}\frac{M}{M_{\rm Pl}} = 0. \tag{6}$$

Imposing the usual $1/r$ potenatial at infinity fixes the physical solution,

$$\pi' = \frac{1}{2}\left(-\frac{\Lambda^3}{2c}r + \sqrt{\left(\frac{\Lambda^3}{2c}r\right)^2 + \frac{2\Lambda^3}{cr}\frac{M}{M_{\rm Pl}}}\right). \tag{7}$$

Inside the Vainshtein radius $r_* \sim (M/M_{\rm Pl}\Lambda^3)^{1/3}$ on the other hand, $\pi \sim r^{1/2}$.

Small fluctuations

Once again, using the ε - representation of the Galileon, it is trivial to write the expression for the quadratic Lagrangian for small fluctuations $\phi = \pi - \pi_c$ on the given background. Shuffling around derivatives and using the antisymmetric structure of the self-interaction, one obtains,

$$
\begin{aligned}
\mathcal{L}^{(2)} &= -\frac{1}{2}(\partial_\mu \phi)^2 + \frac{c}{2\Lambda^3}\varepsilon_{\mu\alpha\rho\gamma}\varepsilon_{\nu\beta\rho\gamma}\partial_\mu\phi\partial_\nu\phi\partial_\alpha\partial_\beta\pi_c \\
&= -\frac{1}{2}(\partial_\mu \phi)^2 + \frac{c}{\Lambda^3}\left(\partial_\mu\partial_\nu\pi_c\partial_\mu\phi\partial_\nu\phi - (\Box\pi_c)^2(\partial_\mu\phi)^2\right) \\
&= \frac{1}{2}\left((\partial_t\phi)^2 - (\partial_r\phi)^2\right) + \frac{c}{\Lambda^3}\left(\left(\pi_c'' + \frac{2}{r}\pi_c'\right)(\partial_t\phi)^2 - \frac{2\pi_c'}{r}(\partial_r\phi)^2\right) \\
&= \left(\frac{1}{2} + \frac{c}{\Lambda^3}\left(\pi_c'' + \frac{2}{r}\pi_c'\right)\right)(\partial_t\phi)^2 - \left(\frac{1}{2} + \frac{2c}{\Lambda^3}\frac{\pi_c'}{r}\right)(\partial_r\phi)^2. \qquad (8)
\end{aligned}
$$

The speed of the fluctuations is given by the ratio of the coefficients of $(\partial_r\phi)^2$ and $(\partial_t\phi)^2$ terms respectively. It is easy to check, that on the solution (7), it exceeds 1 everywhere, leading to superluminality on the given background.