1.1. 

a) \( \Box \phi = \frac{1}{\sqrt{g}} \partial_{\mu} (g^{\mu \nu} \partial_{\nu} \phi) = 0 \)

\[ \Rightarrow 0 = \partial_{\mu} \left( \frac{1}{\sqrt{g}} \partial_{\mu} \phi - \frac{1}{\eta^2} \partial_{\mu} \phi \right) = \frac{1}{\eta^2} \partial_{\mu}^2 \phi - \frac{2}{\eta^3} \partial_{\mu} \phi - \frac{1}{\eta^2} \partial_{\mu} \phi \]

\[ \Rightarrow \partial_{\mu}^2 \phi - \frac{2}{\eta^3} \partial_{\mu} \phi = -k^2 \phi \quad (\text{Fourier Transformation}) \]

Substitute \( f = (1 + i k l \eta) e^{-il \eta} \) into \( \Box \phi \)

\[ \partial_{\mu}^2 f - \frac{2}{\eta^3} \partial_{\mu} f = (k l^2 - i (k l \eta)) e^{-il \eta} - \frac{2}{\eta^3} (k^2 \eta) e^{-il \eta} \]

\[ = -k l^2 (1 + i (k l \eta)) e^{-il \eta} = -k^2 f \]

b) \[ L = - g^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi = \frac{1}{\eta^2} ((\partial_{\mu} \phi)^2 - (\partial_{\nu} \phi)^2) \]

\[ \tau_0 (x) = \frac{\partial^2 L}{\partial (\partial_{\nu} \phi)} = \frac{1}{\eta^2} (\partial_{\mu} \phi) \]

\[ = \frac{1}{\eta^2} \int d^3 x \left[ a^+ e^{i(k l \cdot x)} e^{-il \eta} + a \bar{e}^{-i(k l \cdot x)} e^{il \eta} + k l^2 \eta \right] e^{-iR \cdot x} \]

Let \( f = A (1 + i k l \eta) e^{-il \eta} \).

Then, \[ L (x', x) = [\phi (x'), \tau_0 (x)] \]

\[ = \int d^3 k \int d^3 k' e^{iK' \cdot x'} e^{-iK \cdot x} \left[ a^+ e^{i(k l \cdot x)} e^{-il \eta} + k l^2 \eta \right] (A a^+ e^{-iK \cdot x} + A a e^{iK \cdot x}) \]

\[ = \int d^3 k \int d^3 k' \frac{k^2}{\eta} \left[ (1 + i k l \eta) e^{i(k l \cdot x)} + (1 - i k l \eta) e^{-i(k l \cdot x)} \right] \left[ a^+, a \right] \]

\[ = \frac{A^2 k^2}{\eta} \left[ (1 + i k l \eta) + (1 - i k l \eta) \right] \]

\[ = - 2 \frac{i A^2 k^2}{\eta} \]

\[ \Rightarrow A = \frac{1}{\sqrt{2}} \frac{k \eta}{k^2} \]
Problem Sets for dS/CFT (R. Maldacena).  

1.1 Review of the 2-point function computation in dS. (sequel)

c) The Bunch-Davies vacuum $|BD\rangle$ is chosen such that $a|BD\rangle = 0$ for $m \to -\infty$ (very early times) in the dS space.

In fact, we have $-m^2 + x^2 = N^2$ in dS space, where $N$ is the radius of curvature, thus two observers separated by a small distance ($x^2 \ll m^2$) see $N \approx |\eta|$ and so a huge curvature (asymptotically infinite). The dS space is hence well approximated by the Minkowski space $M_4$ and it is adapted to take the Bunch-Davies vacuum in this way.

The 2-point function at early times $m$ in momentum's space is given by:

$$\langle BD | \phi_R^{\dagger}(\eta) \phi_R(\eta) | BD \rangle$$

$$= |BD| (f_R^2 a_R^{\dagger} + f^*_R a_R) (f^*_R a_R^{\dagger} + f_R a_R^2) |BD\rangle$$

$$= 0 \quad \text{for early times.}$$

$$= |f_R| \frac{1}{N^{3/2}}$$

where $f_R^2$ have been computed in $t$.

$$f_R^2 = \frac{1}{N^{3/2}} (1 + i \sqrt{N}) e^{-i \sqrt{N} t}.$$ 

$$= \langle BD | \phi_R^{\dagger}(\eta) \phi_R(\eta) | BD \rangle = \frac{1}{2N^3} \left(1 + N^2 m^2 \right)$$

which is of order $\frac{1}{N^3}$ for early times.

This relation is true for all $R$ inside the horizon i.e. $R < m^2 \implies R < H^{-1}$ where $H = \frac{\dot{a}}{a}$ is the Hubble radius of the dS space.
d) The Fourier transform of \( \langle BDI \phi_\mathbf{k}^* \phi \mathbf{k} \rangle_{\text{BDI}} \) is:
\[
\langle BDI \phi_\mathbf{k}^* \phi \mathbf{k} \rangle_{\text{BDI}} = \int \frac{d^3p}{(2\pi)^3} \langle BDI \phi_\mathbf{p}^* \phi \mathbf{p} \rangle_{\text{BDI}} e^{i \mathbf{p} \cdot \mathbf{x}}.
\]
\[
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\hbar^3} \left( 1 + \hbar^2 \frac{\eta^2}{p^2} \right) e^{i \mathbf{p} \cdot \mathbf{x}}.
\]
\[
= \frac{1}{(2\pi)^2} \int \frac{dk}{k} \int_0^1 \frac{dk}{k} \cos \left( \frac{\eta^2}{k} \right) e^{ikx}.
\]
which is not so useful.

The second term can be computed:
\[
\frac{1}{(2\pi)^2} \int \frac{dk}{k} \frac{\hbar^2 \eta^2}{k} \sin \left( \frac{\eta^2}{k} \right) = \frac{\eta^2}{2\pi} \int_0^\infty \frac{dk}{kr} \left( e^{i\eta r} - e^{-i\eta r} \right)
\]
\[
= \frac{\eta^2}{2\pi} \frac{1}{r^2}.
\]

The first term is IR divergent.

This two terms are dS-invariant because
\( 1 + \hbar^2 \frac{\eta^2}{p^2} \) is dS-invariant (under re-scaling, notations and spatial translations). So this 2pt function is dS-invariant.

To come back on the divergence, we can say that the absence of mass for the scalar field and the dS spacetime geometry are the physical origin of it.
The 3-point function is: (set $\eta=0$)

$$\left< \psi_{k_1}(\eta=0) \psi_{k_2}(\eta=0) \psi_{k_3}(\eta=0) \right> =$$

$$\langle 0 | T \left( 1 - i \int_{-\infty}^{0} d\eta'' H_I(\eta'') \psi_{k_1}^{\dagger} \psi_{k_2} \psi_{k_3} T \left( -i \int_{0}^{\infty} d\eta'' H_I(\eta'') \right) 0 \right) \rangle$$

$$= \langle 0 | T \left( 1 - i \int_{-\infty}^{0} d\eta'' \left[ \psi_{k_1}^{\dagger} \psi_{k_2} \psi_{k_3}, H_I(\eta'') \right] \right) 0 \rangle$$

Compute $H_I(\eta')$ in Fourier space: $H_I = \int \frac{d^3k}{(2\pi)^3} H_I(\eta')$.

$$H_I(\eta') = \frac{M}{6} \int \frac{d^3p_1}{(2\pi)^3} e^{i p_1 \cdot x} \psi_{p_1}(\eta') \int \frac{d^3p_2}{(2\pi)^3} e^{i p_2 \cdot x} \psi_{p_2}(\eta') \int \frac{d^3p_3}{(2\pi)^3} e^{i p_3 \cdot x} \psi_{p_3}(\eta') \left( \frac{-i}{H\eta'} \right)$$

$$= \frac{M}{6} \int \frac{d^3p_1}{(2\pi)^3} \int \frac{d^3p_2}{(2\pi)^3} \int \frac{d^3p_3}{(2\pi)^3} \psi_{p_1}(\eta') \psi_{p_2}(\eta') \psi_{p_3}(\eta') \left( \delta^{(3)}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \right) \left( \frac{-i}{H\eta'} \right)$$

$$= \frac{1}{6} \int \frac{d^3p_1}{(2\pi)^3} \int \frac{d^3p_2}{(2\pi)^3} \int \frac{d^3p_3}{(2\pi)^3} \psi_{p_1}(\eta') \psi_{p_2}(\eta') \psi_{p_3}(\eta') \left( \delta^{(3)}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \right) \left( \frac{-i}{H\eta'} \right)$$

$$\Rightarrow \left< \psi_{k_1} \psi_{k_2} \psi_{k_3} \right> = -\frac{i}{6} \int d\eta'' \left( \frac{1}{16} \int \frac{d^3p_1}{(2\pi)^3} \int \frac{d^3p_2}{(2\pi)^3} \int \frac{d^3p_3}{(2\pi)^3} \psi_{p_1}(\eta'') \psi_{p_2}(\eta'') \psi_{p_3}(\eta'') \left( \delta^{(3)}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \right) \left( \frac{-i}{H\eta''} \right) \right)$$
First term:
\[ \langle 0 | \Phi_k (\phi) \Phi_{k_2} (\psi) \Phi_{k_3} (\chi) \Phi_{p_1} (\eta^-) \Phi_{p_2} (\eta^-) \Phi_{p_3} (\eta^-)|0 \rangle \]

To obtain fully connected diagrams, only contract \( k_i \)'s with \( p_i \)'s.

\[ \Rightarrow = \langle \Phi_k (\phi) \Phi_{p_1} (\eta^-) \rangle \langle \Phi_{k_2} (\phi) \Phi_{p_2} (\eta^-) \rangle \langle \Phi_{k_3} (\phi) \Phi_{p_3} (\eta^-) \rangle + \text{5 permutations} \]

\[ = \frac{H^2}{2k_i} \delta^3(k_i - p_i) \left( 1 + i p_i \gamma \right) e^{i p_i \eta^2} \frac{H^2}{2k_2} \delta^3(k_2 - p_2) \]

\[ \times \left( 1 - i p_2 \gamma \right) e^{i p_2 \eta^2} \frac{H^2}{2k_3} \delta^3(k_3 - p_3) e^{i p_3 \eta^2} + \text{permute} \]

Plug back into above expression. The 5 permutations give similar contributions. The reverse terms in the commutator are complex conjugate of the term above.

\[ \langle \Phi_k (\phi) \Phi_{k_2} (\psi) \Phi_{k_3} (\chi) \rangle = -i M \int_0^\infty d\eta^2 \frac{(2\pi)^3 \delta^3(k_i + k_2 + k_3)}{-H \eta^2} \frac{H^6}{8k_i k_2 k_3} \]

\[ (1 - i k_1 \gamma)(1 - i k_3 \gamma)(1 - i k_2 \gamma) e^{i \Phi_k + \frac{1}{2} \Phi_{k_2} + \frac{1}{2} \Phi_{k_3} \eta^2} + \text{h.c.} \]
\begin{align*}
\langle \phi_{k_1}, \phi_{k_2}, \phi_{k_3} \rangle &= -\frac{\delta M}{8k_1^3 k_2^3 k_3^3} \frac{(2\pi)^3}{\delta (k_1 + k_2 + k_3)} \\
&\times \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_3}{(2\pi)^3} \frac{(i\hbar)^3}{\delta (k_1 + k_2 + k_3)} e^{i(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3} \cdot \mathbf{r})} + h.c.
\end{align*}

= (2\pi)^3 \frac{\delta^3 \delta (k_1 + k_2 + k_3)}{8k_1^3 k_2^3 k_3^3} \frac{2}{\delta (k_1 + k_2 + k_3)} \left( \sum_{\text{c}} i k_1^2 + 4 \sum_{i,j} \frac{k_i k_j}{\delta_{ij}} \right)

+ \text{I.I.}

Using numerical integration
de Sitter hyperboloid:

\[ -x_0^2 + \sum_{i=1}^{4} x_i^2 = a^2 \]  \quad \text{(embedded in } \mathbb{R}^{4,1})

For flat slicing, define

\[ x_0 = x \sinh (\frac{t}{\alpha}) + \frac{r^2 e^{t/\alpha}}{2\alpha} \]

\[ x_i = x \cosh (\frac{t}{\alpha}) - \frac{r^2 e^{t/\alpha}}{2\alpha} \]

\[ x_i = e^{t/\alpha} y_i, \quad i = 2, 3, 4 \]

\[ r^2 = y_2^2 + y_3^2 + y_4^2 \]

\[ \Rightarrow \text{plug these in and we get} \]

\[ ds^2 = -dt^2 + e^{2t/\alpha} \sum_{i} dy_i^2 \]

\[ \text{flat space metric} \]

2.7 de Sitter in 4d has isometry group O(4,1) with 10 generators: 3 rotations, 3 translations, 3 "boosts", 1 scaling symmetry

We evaluate correlation function \( G(\mathcal{M}_0) \) at a particular point in time, \( \mathcal{M}_0 \). When we do this, we lose the isometries involving time because we pick out a specific point in time to evaluate the correlations. So we are only left with the 3 rotations, translations, + scale invariance – 7 symmetries.
(* Group 18 *)

(* The system of coupled equations *)

Ωc = 0.95;
Ωm = 1;
Ωb = 0.05;
h = 0.5;
α = Sqrt[21.5 * Ωm h²];
y[x_] := (α x)² + 2 α x;
yc[x_] := y[x] Ωc / Ωm;
yb[x_] := 1.68 y[x] Ωb / Ωm;
η[x_] := 2 α (α x + 1) (α² x² + 2 α x);
x0 = 0.001;
x0 = 0.001;
y0 := (α * x0)² + 2 α * x0;
η0 := 2 α (α x0 + 1) / (α² x0² + 2 α x0);
Syst :=
{D[δc[x], x] == -κ vc[x] + 3 D[φ[x], x],
 D[δy[x], x] == -δy[φ[x]] + 4 D[φ[x], x],
 D[vc[x], x] == -η[x] vc[x] + κ φ[x],
 D[vy[x], x] == -η[y] yb[x] vy[x] + κ δy[x] / 3
   + 4 / 3 + yb[x],
 D[φ[x], x] == -η[x] φ[x] + (3 η[x]² (vy[x] (4 / 3 + y[x] + yc[x]) + yc[x] vc[x]))
   + 4 / 3 + yb[x],
 δc[x0] := 3 / 4 δy[x0],
 δy[x0] := -2 (1 + 3 y0 / 16),
 vc[x0] := κ (δy[x0] + 2 κ² (1 + y0) / 9 η₀² (4 / 3 + y0)),
 vy[x0] := vc[x0],
 φ[x0] := 1
};

(* A definition for the dots in the plots -- not very pretty but necessary *)
solution =
{(0, 0, 0, 0, 0), (0, 0, 0, 0, 0), ... and so on ..., (0, 0, 0, 0, 0), (0, 0, 0, 0, 0)};
(* At time x=0.1 *)
For[i = 1, i < 721, i++,
{
  sol = NDSolve[ Sys / . x -> i/3, \[Delta]c, \[Delta]y, vc, vy, \[Phi], \{x, x0, 1\}],
  solution[i, 1] = Evaluate[\[Delta]c[0.1] /. sol][1],
  solution[i, 2] = Evaluate[\[Delta]y[0.1] /. sol][1],
  solution[i, 3] = Evaluate[vc[0.1] /. sol][1],
  solution[i, 4] = Evaluate[vy[0.1] /. sol][1],
  solution[i, 5] = Evaluate[\[Phi][0.1] /. sol][1]
}

{ListPlot[Table[{Log[n/1.5], solution[n][1]}, \{n, 1, 720, 1\}]*JoinedTrue\}]
, ListPlot[Table[{Log[n/1.5], solution[n][2]}, \{n, 1, 720, 1\}]*JoinedTrue\}]
, ListPlot[Table[{Log[n/1.5], solution[n][3]}, \{n, 1, 720, 1\}]*JoinedTrue\}]
, ListPlot[Table[{Log[n/1.5], solution[n][4]}, \{n, 1, 720, 1\}]*JoinedTrue\}]
, ListPlot[Table[{Log[n/1.5], solution[n][5]}, \{n, 1, 720, 1\}]*JoinedTrue\}]

(*At time x=0.5*)
For[i = 1, i < 721, i++,
{
   sol = NDSolve[Syst /. x → i / 3, {δc, δy, vc, vy, φ}, {x, x0, 1}],
   solution[i, 1] = Evaluate[δc[0.5] /. sol][1],
   solution[i, 2] = Evaluate[δy[0.5] /. sol][1],
   solution[i, 3] = Evaluate[vc[0.5] /. sol][1],
   solution[i, 4] = Evaluate[vy[0.5] /. sol][1],
   solution[i, 5] = Evaluate[φ[0.5] /. sol][1]
   }
]
{ListPlot[Table[{Log[n / 1.5], solution[n][1]}, {n, 1, 720, 1}]*, Joined->True]],
ListPlot[Table[{Log[n / 1.5], solution[n][2]}, {n, 1, 720, 1}]*, Joined->True]],
ListPlot[Table[{Log[n / 1.5], solution[n][3]}, {n, 1, 720, 1}]*, Joined->True]],
ListPlot[Table[{Log[n / 1.5], solution[n][4]}, {n, 1, 720, 1}]*, Joined->True]],
ListPlot[Table[{Log[n / 1.5], solution[n][5]}, {n, 1, 720, 1}]*, Joined->True]]
Group 18.

x) CMB and LSS (Professor Aldarriaga).

Question 12.

From Ma & Bertschinger (95), we have the following dynamical equations for the density field and the divergence of the velocity field:

\[ \frac{\delta}{\Delta t} \left( 1 + \omega \right) (\delta - 3 \phi') - \frac{3 \omega}{a} \left( \frac{\delta \rho}{\delta} - \omega \right) S = \]

\[ \frac{g'}{a} \left( 1 - 3 \omega \right) \theta - \omega' \theta + \frac{2 \omega}{1 + \omega} \left( \frac{\delta \rho}{\delta} - \omega \right) \]

These equations are valid for the met (mass averaged) fields \( \delta_c, \delta_v \) and \( \delta_c, \delta_v \) of CDM and photons \( \phi \).

In fact, because the two fluids interact by Thomson scattering, we need to add a term in the \( \delta \) equation:

\[ \frac{4}{3} \pi a \left( 1 - \omega \right) \left( \delta_c - \delta_v \right) \]

with \( \epsilon / c = 0.6652 \times 10^{-24} \) cm² and \( \omega \) the number density of electrons.

(I) and (II) are written in the conformal Newtonian gauge.

x) For CDM we obtain, with \( \omega_c = 0 \) and \( \delta_c = k \rho \).

In Fourier space:

\[ \delta_c' = - (k \rho - 3 \phi') - 3 H (\omega - \omega_c) \delta \]

\[ = - k \rho + 3 \phi \]

with 'dotted' \( 2/\Delta t \) (conformal time).

Noting \( (\cdots) = (\cdots) - \int_{\cdots}^{(\cdots)} \frac{dx}{x} \) with \( x = \frac{t}{\Delta t} \)

we thus have:

\[ \delta_c = - k \rho + 3 \phi \]

with \( k = k \rho > 0 \).

\[ k \rho c' = \frac{\delta c}{a} \frac{\rho c}{1 + \omega} \]

\[ \omega_c = 0 \quad \text{and} \quad \rho c^2 \frac{d \rho}{d \theta} = 0 \quad \text{so} \quad \psi = \phi \]
Robustness of GR (Anhari-Hamed)

**Question 4)**

Let us start with the action:

\[ L = \int d^4 x \sqrt{-g} \left[ F(A)(\nabla^2 A) + F(A) \right] \text{ where } A = A(x). \]

The equation of motion for \( A \) is given by:

\[ \frac{\partial L}{\partial A} = 0 \quad \Rightarrow \quad F''(A)(\nabla^2 A) = F'(A) + F(A) \]

so we must have:

(i) either \( A = \nabla \)

(ii) or \( F''(A) = 0 \).

Both these cases lead to the conclusion that \( S \) can be written as:

\[ S = \int d^4 x \sqrt{-g} F(\nabla). \]

**Proof**: (i) \( A = \nabla \Rightarrow S = \int d^4 x \sqrt{-g} \left[ 0 + F(A = \nabla) \right] \checkmark \)

(ii) \( F''(A) = 0 \Rightarrow F(A) = C_1 A + C_2 \) with \( C_1, C_2 \) two constants.

Then:

\[ S = \int d^4 x \sqrt{-g} \left[ C_1 (\nabla - A) + C_1 A + C_2 \right] \]

\[ = \int d^4 x \sqrt{-g} \left[ C_1 \nabla + C_2 \right] = \int d^4 x \sqrt{-g} F(\nabla) \checkmark. \]

Let us now do a conformal transformation to demix \( A(x) \) from the metric.

We take \( g_{ab} \rightarrow \tilde{g}_{ab} = g_{ab} \cdot \sigma^2 \) and so we have

\[ \sqrt{-g} = \sqrt{-\tilde{g}} \sigma^{-4}. \]

The scalar curvature transform as:

\[ R \rightarrow \tilde{R} = \sigma^{-2} \left[ R + 6 \frac{\tilde{\nabla}^2 \sigma}{\sigma} \right] \]

and hence \( \tilde{R} = \sigma^{-2} \left[ \tilde{R} - 6 \frac{\tilde{\nabla}^2 \sigma}{\sigma} \right] \).

Thus we have:

\[ S = \int d^4 x \sqrt{-\tilde{g}} \sigma^{-4} \left[ F(A) \cdot \sigma^2 \left( \tilde{R} - 6 \frac{\tilde{\nabla} \tilde{\nabla} \sigma}{\sigma} \right) - F'(A) \cdot A + F(A) \right] \]
We choose $\Sigma$ such that $F'(A) = \delta^2$.

$$S' = \int d^4x \sqrt{g} \left[ \tilde{R} - \frac{6}{\sqrt{g}} \nabla^a F'(A)^{3/2} - \frac{F'(A)}{F'(A)^2} \frac{F'(A) A + F(A)}{F(A)} \right]$$

Defining $\chi(x) = -\log F'(A)$ (i.e., $F'(A) = e^{-\chi}$), we have:

$$\nabla^a \nabla_a e^{-\chi/2} = -3 e^{-\chi/2} \frac{\tilde{\nabla} \tilde{\nabla} (\partial \phi \cdot e^{-\chi/2})}{e^{-\chi/2}}$$

$$= -3 e^{-\chi/2} \left[ \tilde{\nabla} \tilde{\nabla} (\partial \phi \cdot e^{-\chi/2}) - \frac{\tilde{\nabla}^2 (\partial \phi \cdot e^{-\chi/2})}{2} \right]$$

$$= -3 \frac{\tilde{\nabla} \tilde{\nabla} (\partial \phi)}{e^{-\chi/2}} + 3 (\partial \chi)^2.$$  

But the first term in the $\nabla^2$ is a total divergence and thus don't contribute to the action.

Defining $V(\chi) = \frac{A}{F(A)} - \frac{F(A)}{F'(A)^2}$, we have:

For final result:

$$S = \int d^4x \sqrt{g} \left[ \tilde{R} - \frac{3}{2} \nabla^a \nabla_a e^{-\chi/2} - V(\chi) \right]$$

Thus we have shown that the $F(R)$-gravity action was equivalent to a scalar field coupled to Einstein gravity.