Group 1

MALDACENA - 1

1.1) Maldacena

Metric $ds^2 = -\frac{d\eta^2 + dx^2}{\eta^2}$

a) $S = \int d^4x \sqrt{-g}\, (\nabla\phi)^2 = \int d\eta \frac{1}{\eta^2}\left((\partial_\eta \phi)^2 - (\nabla_i \phi)^2 \right)$

$\partial_\eta \left(\frac{1}{\eta^2} \dot\phi \right) - \partial_i \left(\frac{1}{\eta^2} \partial_i \phi \right) = 0$

$\Rightarrow \ddot\phi - \frac{3}{\eta} \dot\phi - \nabla^2 \phi = 0$ Wave equation in dS_4

In Fourier $\phi(\vec{x}, \eta) = \int d^3\vec{k}\, f(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}$

$\Rightarrow \boxed{\ddot f - \frac{2}{\eta} \dot f + k^2 f = 0}$

If $f \sim (1 + i|\vec{k}|\eta) e^{-i|\vec{k}|\eta}$

$\dot f \sim k^2 \eta e^{-ik\eta}$

$\ddot f \sim (k^2 - ik^3\eta) e^{-ik\eta}$

$\left[(\cancel{k^2} - \cancel{ik^3\eta}) - \frac{2}{\eta} \cancel{k^2}\eta + \cancel{k^2}(1 + \cancel{ik\eta}) \right] e^{-ik\eta} = 0$ solves the equation

EOM is real then f^* will solve it too.

b) Let $\phi = a^\dagger f + a f^\dagger$, $\Pi_\phi = \frac{\partial \mathcal{L}}{\partial(\partial_\eta \phi)} = \frac{1}{\eta^2} \dot\phi$

$f = A(1 + ik\eta) e^{-ik\eta}$

$= \frac{1}{\eta^2}\left[a^\dagger \dot f + a \dot f^\dagger \right]$

Demanding

$$[\phi_{\vec{k}}, \pi_{\vec{k}'}] = i\ \delta(\vec{k}+\vec{k}')$$

And $[a, a] = 1$

$$[\phi_{\vec{k}}, \pi_{\vec{k}'}] = \frac{1}{\eta^2}[a^\dagger f + a f^\dagger,\ a^\dagger \dot{f} + a \dot{f}^\dagger]$$

$$= \frac{1}{\eta^2}(f^\dagger \dot{f} - \dot{f} f^\dagger) \times \delta(\vec{k}+\vec{k}') = \frac{|A|^2}{\eta^2}(2ik^3\eta^2)\ \delta(\vec{k}+\vec{k}')$$

$$= i\,\delta(\vec{k}+\vec{k}') \Rightarrow |A| = \frac{1}{\sqrt{2k^3}}$$

$$\Rightarrow f = \frac{1}{\sqrt{2k^3}}(1+ik\eta)e^{-ik\eta}$$

c)

$$\langle BD|\phi_{\vec{k}}(\eta)\,\phi_{\vec{k}'}(\eta)|BD\rangle = \langle BD|(a^\dagger f + a f^\dagger)(a^\dagger f + a f^\dagger)|BD\rangle$$

$$= (2\pi)^3\delta^3(\vec{k}+\vec{k}')\,|f|^2$$

$$= (2\pi)^3\delta^3(\vec{k}+\vec{k}')\,\frac{1}{2k^3}(1+k^2\eta^2)$$

d) Summing over all modes $\int \frac{d^3\vec{k}\,d^3\vec{k}'}{(2\pi)^3}\, e^{i\vec{k}\cdot x}\, e^{i\vec{k}'\cdot\vec{x}'}$

$$\Rightarrow \langle BD|\phi(\vec{x},\eta)\,\phi(\vec{x}',\eta)|BD\rangle = \int d^3\vec{k}\,\frac{1}{2k^3}(1+k^2\eta^2)\,e^{i\vec{k}(\vec{x}-\vec{x}')}$$

Diverges !!

CREMINELLI - 6

Group. 1 ①

Creminelli's problem 6

Define a quantum field theory on (fixed) deSitter space,

whose metric is (flat - slicing)

$$ds^2 = \frac{1}{\eta^2}(-d\eta^2 + dx^2) \qquad \sqrt{-g} = \frac{1}{\eta^4}$$

The action is

$$S = \int \frac{d\eta\, d^3x}{H^2}\left[\frac{1}{2\eta^2}\left[\frac{\partial}{\partial\eta}\phi\right]^2 - \frac{1}{2\eta^2}(\nabla\phi)^2 + \frac{g}{6\eta^4}\phi^3(x,\eta)\right]$$

Our goal is to calculate the 3-pt correlator.

$$\langle\Omega|\phi_{k_1}(\eta)\phi_{k_2}(\eta)\phi_{k_3}(\eta)|\Omega\rangle$$

① Recall for the free theory,

$$S = \int d\eta\, dx^3\left[\frac{1}{\eta^2}\left[\frac{\partial}{\partial\eta}\phi\right]^2 - \frac{1}{\eta^2}(\nabla\phi)^2\right]$$

$$\phi(\vec{k},\eta) = f(\vec{k},\eta)a^\dagger_{\vec{k}} + f^*(\vec{k},\eta)a_{-\vec{k}}$$

where $f(k,\eta) = \frac{H}{\sqrt{2k^3}}(1-ik\eta)e^{ik\eta}$, $k = |\vec{k}| \geq 0$.

So that ϕ & π (or equivalently a & $a^\dagger$)

Satisfy the canonical commutation relation.

$$[a_k, a^\dagger_{k'}] = \delta^{(3)}(\vec{k}+\vec{k}')(2\pi)^3$$

$$[\pi(k,\eta), \phi(k',\eta)] = -i\delta^{(3)}(k+k')(2\pi)^3$$

②

and the two-pt correlator.

$$\langle \phi(k,\eta)\,\phi(k',\eta)\rangle = \frac{H^2}{2k^3}\,\delta(\vec{k}+\vec{k}')\,(1+k\eta^2)$$

BD BD

Especially.

$$\langle BD|\,\phi(k,\eta)\,\phi(k',\eta')\,|BD\rangle$$

$$= \langle BD|\left[f(k,\eta)\,a^\dagger_{\vec{k}} + f^*(k,\eta)\,a_{-\vec{k}}\right]\left[f(k',\eta')\,a^\dagger_{\vec{k}'} + f^*(k',\eta')\,a_{-\vec{k}'}\right]|BD\rangle$$

$$= f^*(k,\eta)\,f(k',\eta')\,\delta^{(3)}(\vec{k}+\vec{k}')$$

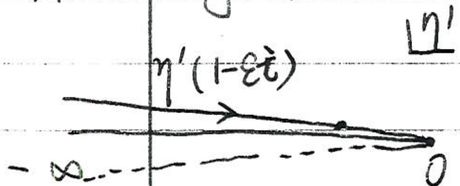
② Now let's come to compute $\langle \phi_{k_1}\phi(k_2)\phi(k_3)\rangle$

To the first order in g,

$$\langle \Omega|\,\phi(k_1,\eta)\,\phi(k_2,\eta)\,\phi(k_3,\eta)\,|\Omega\rangle$$

$$= -i\int_C^{\eta} d\eta'\,\langle BD|\left[\phi(k_1,\eta)\,\phi(k_2,\eta)\,\phi(k_3,\eta),\ H_{int}(\eta')\right]|BD\rangle$$

The integral contour is chosen to be.



basically it is a technique to project the interacting vacuum to the well defined BD vacuum;

because of ~~having~~ taken [ ] already, need to go over half of the in-in contour.

@ Compute the commutator $\langle [\phi_I(k_1,\eta)\phi_I(k_2,\eta)\phi_I(k_3,\eta),\ H^I_{int}(\eta')]\rangle$

$$H^I_{int}(\eta') = \int d^3x \frac{g}{6}\phi_I^3(x,\eta')\frac{1}{\eta'^4}$$

$$= \frac{g}{6}\int d^3x \int \prod_{i=1}^{3}\frac{d^3k_i}{(2\pi)^3} e^{ik_i x}\phi_I(k_1,\eta)\phi_I(k_2,\eta)\phi_I(k_3,\eta)\frac{1}{\eta'^4}$$

$$= \frac{g}{6}\int \prod_{i=1}^{3}\frac{dq_i}{(2\pi)^3}\frac{1}{\eta'^4}\phi_I(q_1,\eta')\phi_I(q_2,\eta')\phi_I(q_3,\eta')(2\pi)^3\delta^{(3)}(\vec{q}_1+\vec{q}_2+\vec{q}_3)$$

$$\langle BD|\phi_I(k_1,\eta)\phi_I(k_2,\eta)\phi_I(k_3,\eta)\phi_I(q_1,\eta')\phi_I(q_2,\eta')\phi_I(q_3,\eta')|BD\rangle$$

$$= \langle BD| + - - + - + |BD\rangle + \text{permutations}$$

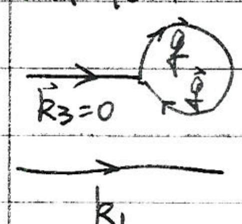
$C_6^3 = \frac{6\times5\times4}{3!} = 20$ in total.

positive mode   negative mode.

If there are too many " −'s on the right, for instance.

$\langle + + - - - + \rangle$, they all vanish.

Furthermore, we should choose positive mode for $\phi_I(q_i,\eta')$, otherwise contraction between $\phi_I(q_i,\eta')$ & $\phi(q_j,\eta')$ will lead to diagrams like.



which is infinite & can be cancelled by counterterms.

$$\because \langle BD | [\phi_I(k_1,\eta)\phi_I(k_2,\eta)\phi_I(k_3,\eta), \phi_I(q_1,\eta')\phi_I(q_2,\eta')\phi_I(q_3,\eta') | BD\rangle$$

$$\cancel{= \langle BD | a(-k_1,\eta) a(-k_2,\eta) a(-k_3,\eta) a^\dagger(q_1,\eta') a(q_2,\eta'}$$

$$= \langle BD | a(-\vec{k}_1) a(-\vec{k}_2) a(-\vec{k}_3) a^\dagger(\vec{q}_1) a^\dagger(\vec{q}_2) a^\dagger(\vec{q}_3) | BD\rangle \prod_{i=1}^{3} f^*(-\vec{k}_i,\eta) f(\vec{q}_i,\eta')$$

$$- \langle BD | a(-\vec{q}_1) a(-\vec{q}_2) a(-\vec{q}_3) a^\dagger(\vec{k}_1) a^\dagger(\vec{k}_2) a^\dagger(\vec{k}_3) | BD\rangle \cdot \prod_{i=1}^{3} f^*(-\vec{q}_i,\eta) f(\vec{k}_i,\eta)$$

$$= \left[\delta^{(3)}(\vec{q}_1+\vec{k}_1)\delta^{(3)}(\vec{q}_2+\vec{k}_2)\delta^{(3)}(\vec{q}_3+\vec{k}_3) \prod_{i=1}^{3} f^*(\vec{k}_i,\eta) f(\vec{q}_i,\eta') (2\pi)^9 - c.c. \right] + 5 \text{ permutation}.$$

$$\therefore \langle \Omega | \phi(k_1,\eta)\phi(k_2,\eta)\phi(k_3,\eta) | \Omega \rangle$$

$$= \left[-i \int_C \frac{d\eta'}{\eta'^4} \frac{g}{6} (2\pi)^3 \delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3) \prod_{i=1}^{3} f^*(\vec{k}_i,\eta) f(\vec{k}_i,\eta') + c.c. \right]$$

$$+ 5 \text{ permutation}.$$

$$= \frac{g}{6} \int_{-\infty}^{\eta} \frac{d\eta'}{\eta'^4} (2\pi)^3 \delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)(-i) \frac{H^6}{(2k_1^3)(2k_2^3)(2k_3^3)} (1+ik_1\eta)(1+ik_2\eta)(1+ik_3\eta)$$

$$e^{-ik_t\eta}(1-ik_1\eta')(1-ik_2\eta')(1-ik_3\eta') e^{ik_t\eta'(1-i\varepsilon)}$$

$$+ c.c$$

EulerGamma

using Mathematica.

⑤

$$= g(2\pi)^3 \delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{H^6}{(2k_1^3)(2k_2^3)(2k_3^3)}\cdot\frac{2}{3}\Big[(\gamma-1)(k_1^3+k_2^3+k_3^3)$$

$$-(k_1^2k_2+k_1^2k_3+k_1k_2^2+k_2^2k_3+k_1k_3^2+k_2k_3^2)+k_1k_2k_3$$

$$+(k_1^3+k_2^3+k_3^3)\ln(k_t|\eta|)\Big]+O(\eta)$$

where we have assumed $\eta \to 0^-$. and only keep the leading term.

CREMINELLI - 7

Group 1.

Creminelli's Problem 7.

The isometry group of the deSitter space is $SO(4,1)$.

They are:

$$ds^2 = \frac{-d\eta^2 + d\vec{x}^2}{\eta^2}$$

3 spatial translation. $x^i \to x^i + a^i$

3 spatial rotation $x^i \to x'^i = R^{ij} x^j$,
$R^{ij} \in SO(3)$

1 dilatation. $x^i \to x'^i = \lambda x^i$
$\eta \to \eta' = \lambda \eta$

3 special conformal transformation (infinitesmally)
$$x^i \to x^i + 2(\vec{x}\cdot\vec{b})x^i - b^i(-\eta^2 + \vec{x}^2)$$
$$\eta \to \eta + 2(\vec{x}\cdot\vec{b})\eta$$

(1) If consider a field theory defined in a fixed deSitter backgroupd which is deSitter invariant.
The correlation functions would respect all the 10 symmetries.

(2) In slow-rolling inflation model, the correlation function of ζ only preserve the symmetries only involving spatial transformations (spatial translation & rotations) rather than the ones mixing space & time.

It could be understood in the sense that even if we start from a deSitter space and assume that the backreaction is small, since we have chosen a time-dependent background solution $\phi_0(\eta)$,

thus the ~~I~~ correlator will not preserve scale invariance & SCT invariance.

In the slow-rolling model, the extent of symmetry breaking is controlled by the small-rolling parameters in the sense of which these symmetries are just weakly broken.

GROUP-1

IUCAA

IAS HOMEWORK-1 $\boxed{\text{SUSSKIND - Q1}}$

Q:1: Derive the metric of de Sitter space in the flat slicing.

DS METRIC : $ds^2 = -dX_0^2 + \sum_{i=1}^{4} dX_i^2 \longrightarrow (\mathrm{I})$

CONSTRAINT : $R^2 = -X_0^2 + \sum_{i=1}^{4} X_i^2 \longrightarrow (\mathrm{II})$

Perform a coordinate transformation:

$$\begin{cases} X_0 = R\left[\sinh\tau + \frac{r^2 e^{\tau}}{2}\right] \\ X_1 = R\left[\cosh\tau - \frac{r^2 e^{\tau}}{2}\right] \\ X_i = Re^{\tau} y_i \quad (\text{where } i = 2,3,4) \end{cases}$$

Also $\sum_{i=2}^{4} y_i^2 = r^2 \Rightarrow y_2\,dy_2 + y_3\,dy_3 + y_4\,dy_4 = 0$

$\longrightarrow (\mathrm{III}) \qquad \longrightarrow (\mathrm{IV})$

So, $dX_0 = R\cosh\tau\,d\tau + \frac{Rr^2e^{\tau}}{2}d\tau$

$dX_1 = R\sinh\tau\,d\tau - \frac{Rr^2e^{\tau}}{2}d\tau$

$dX_i = Re^{\tau}y_i\,d\tau + Re^{\tau}dy_i$

Thus equation (I) becomes: $-R^2\cosh^2\tau\,d\tau^2 - \frac{R^2r^4}{4}e^{2\tau}d\tau^2 - \frac{R^2r^2}{1}e^{\tau}\cosh\tau\,d\tau^2$

$+R^2\sinh^2\tau\,d\tau^2 + \frac{R^2r^4}{4}e^{2\tau}d\tau^2 - R^2r^2e^{\tau}\sinh\tau\,d\tau^2$

$+\sum_{i=2}^{4}R^2e^{2\tau}y_i^2\,d\tau^2 + \sum_{i=2}^{4}R^2e^{2\tau}dy_i^2 + \sum_{i=2}^{4}R^2e^{2\tau}d\tau\,y_i\,dy_i$

$\Rightarrow ds^2 = -R^2d\tau^2[\cosh^2\tau - \sinh^2\tau] - R^2r^2e^{\tau}(\cosh\tau + \sinh\tau)\,d\tau^2 + R^2e^{2\tau}d\tau^2\sum_{i=2}^{4}y_i^2$

$+R^2e^{2\tau}\sum_{i=2}^{4}dY_i^2 + R^2e^{2\tau}d\tau\sum_{i=2}^{4}y_i\,dY_i$

Using (III) and (IV), we get,

$ds^2 = -R^2d\tau^2 - R^2r^2e^{\tau}e^{\tau}d\tau^2 + r^2e^{2\tau}d\tau^2R^2 + e^{2\tau}R^2\sum_{i=2}^{4}dy_i^2 + 0$

$= -R^2d\tau^2 - R^2r^2e^{2\tau}d\tau^2 + R^2r^2e^{2\tau}d\tau^2 + R^2e^{2\tau}\sum_{i=2}^{4}dY_i^2$

$= R^2\left[-d\tau^2 + e^{2\tau}dY^i dY^i\right]$ where $i = 2,3,4.$

Hence Proved.

4. Mechanisms for Inflation (Eva Silverstein)
4) Warped D-brane slow roll inflation. Estimate tension of cosmic strings produced at the end.

Solution: Consider a warped throat in a string compactificatio[n]. Slow roll inflation happens when a mobile D-brane, such as D3-brane, slowly moves down the throat and eventually annihilates with an anti-D3-brane. When they collide at the end of inflation, various extended objects could be produced, including D1-branes which could becom[e] cosmic strings. More generally, a p-brane wrapping p-1 internal dimensions could become a cosmic string.

Let us first consider D1-branes. Their tension in ten dimensional string frame is given by

$$T_1 = \frac{1}{2\pi g_s \alpha'}, \quad S_{D1} = -\int d^2\xi \; T_1 \sqrt{-\det(g_{ab})}$$

where (ξ^0, ξ^1) are worldvolume coordinates on the D1-brane, g_{ab} is the induced metric with $a, b = 0, 1$.

Suppose the 10D metric is given by

$$ds_{10}^2 = e^{2A(y)} g^{(s)}_{\mu\nu} dx^\mu dx^\nu + g_{ij} dy^i dy^j$$

where $g^{(s)}_{\mu\nu}$ is the 4D string frame metric.

To obtain the tension of the cosmic string as measured in 4D, we need to
① ~~conver~~ take in account warping $e^{2A(y)}$,
② and convert to 4D Einstein frame.

For ①, we write the warp factor at the bottom of the slow-inflati[on] throat as $e^{2A_{min}}$

This is where the brane-antibrane pair collides and reheating happens, producing D1-branes.
For D1-branes to become cosmic string, they must be extended in 2 of the 4D noncompact directions.
This means the induced metric $g_{ab} = e^{2A_{min}} g^{(s)}_{ab}$

so $S_{D1} = -\int d^2\xi \, T_1 e^{2A_{min}} \sqrt{-\det(g^{(s)}_{ab})}$

For ②, converting to 4D Einstein frame is equivalent to writing down the tension in 4D Planck units. This is a useful thing to do (e.g. this determines the deficit angle produced by the cosmic string), so let us do it.

The Einstein action in 10D is

$$S_g = \frac{1}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{-g_{10}}\, e^{-2\phi} R_{10}$$

$$\sqrt{-g_{10}} = e^{4A(y)} \sqrt{-g_4^{(s)}} \sqrt{\det(g_{ij})}\ , \quad R_{10} = e^{-2A(y)} R_4^{(s)} + \cdots$$

Defining the "weighted" volume of compactification as

$$V_W = \int d^6y \sqrt{\det(g_{ij})}\, e^{2A(y)}$$

we get

$$S_g = \frac{1}{(2\pi)^7 \alpha'^4} \int d^4x \sqrt{-g_4^{(s)}}\, \frac{V_W}{g_s^2} R_4^{(s)} + \cdots$$

In order to get the canonical Einstein action in 4D, we define the 4D Einstein metric by

$$g^{(E)}_{\mu\nu} = \frac{2V_W}{(2\pi)^7 \alpha'^4 g_s^2 M_{Pl}^2} g^{(s)}_{\mu\nu}$$

where M_{Pl}, the 4D Planck mass, is determined by the stabilized value of V_W and g_s via the above formula.

The D1-brane action in 4D Einstein frame becomes

$$S_{D1} = -\int d^2\xi \; T_1 e^{2A_{min}} \frac{(2\pi)^7 \alpha'^4 g_s^2 M_{Pl}^2}{2V_W} \sqrt{-\det(g_{ab}^{(E)})}$$

from which we read off the tension in 4D Einstein frame as

$$T_1^{(E)} = T_1 e^{2A_{min}} \frac{(2\pi)^7 \alpha'^4 g_s^2 M_{Pl}^2}{2V_W}$$

In terms of Planck units it is

$$\frac{T_1^{(E)}}{M_{Pl}^2} = \frac{1}{2\pi g_s \alpha'} e^{2A_{min}} \cdot \frac{(2\pi)^7 \alpha'^4 g_s^2}{2V_W}$$

$$\boxed{\frac{T_1^{(E)}}{M_{Pl}^2} = e^{2A_{min}} \cdot \frac{(2\pi)^6 \alpha'^3 g_s}{2V_W}}$$

For a general Dp-brane wrapping a (p-1)-cycle in the internal dimensions, the tension of the cosmic string is

$$\frac{T_p^{(E)}}{M_{Pl}^2} = \frac{V(\Sigma^{p-1})}{(2\pi)^p g_s \alpha'^{(p+1)/2}} e^{2A_{min}} \cdot \frac{(2\pi)^7 \alpha'^4 g_s^2}{2V_W}$$

$$\boxed{\frac{T_p^{(E)}}{M_{Pl}^2} = e^{2A_{min}} \cdot \frac{(2\pi)^{7-p} \alpha'^{(7-p)/2} g_s V(\Sigma^{p-1})}{2V_W}}$$

where $V(\Sigma^{p-1})$ is the physical volume of the (p-1)-cycle wrapped by the Dp-brane.

Group 1, PiTP 2011

6. Robustness of GR. Attempts to modify gravity

4. Verify $F(R)$ gravity is equivalent to

$$\int d^4x\sqrt{-g}\,[F'(A)(R-A)+F(A)]$$

Since A is a non-dynamical auxiliary field, we may use its EOM to eliminate it. The EOM is

$$F''(A)(R-A)-F'(A)+F'(A)=0$$

Assuming $F''(A)\neq 0$, we have $A=R$. Plugging this into the action, we get $\int d^4x\sqrt{-g}\,F(R)$. So they're equivalent.

Now, we need to go to Einstein frame by a Weyl transformation:

$$\tilde{g}_{\mu\nu}=F'(A)\,g_{\mu\nu},$$

Using Eq. (D.9) in Wald with $\Omega=(F'(A))^{-1/2}$, we have

$$R=F'(A)\Big\{\tilde{R}-6\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\left(-\tfrac{1}{2}\ln F'(A)\right)-6\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\left[-\tfrac{1}{2}\ln F'(A)\right]\times\tilde{\nabla}_\nu\left[-\tfrac{1}{2}\ln F'(A)\right]\Big\},\quad \sqrt{-g}=\sqrt{-\tilde{g}}\,\frac{1}{(F'(A))^2}$$

So

$$S=\int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{F'(A)}(R-A)+\frac{F(A)}{F'(A)^2}\right]$$

$$=\int d^4x\sqrt{-\tilde{g}}\left\{\tilde{R}+3\tilde{\nabla}^2\ln F'(A)-\frac{3}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\ln F'(A)\,\tilde{\nabla}_\nu\ln F'(A)-\frac{A}{F'(A)}+\frac{F(A)}{F'(A)^2}\right\}$$

total derivative

Defining $\sigma=-\ln F'(A)$, we have (dropping all tildes)

$$S=\int d^4x\sqrt{-g}\left[R-\frac{3}{2}g^{\mu\nu}\partial_\mu\sigma\,\partial_\nu\sigma-\underbrace{\left(\frac{A}{F'(A)}-\frac{F(A)}{F'(A)^2}\right)}_{=V(\sigma)}\right]$$

QED.

EXTRA CREDIT SOLUTIONS

Maldacena's Problem Set 1.3

1.3 (ab) $S=-\frac{1}{2}\int d^4x\sqrt{-g}\,(\nabla\phi)^2$, $ds^2=\frac{-d\eta^2+d\vec{x}^2}{\eta^2}$

Classical solutions to EOM of ϕ:

$$\phi_{\vec{k}}(\eta)=c_1(1-ik\eta)e^{ik\eta}+c_2(1+ik\eta)e^{-ik\eta}\,, \quad \text{where } k\equiv|\vec{k}|$$

Early time boundary condition: $c_2=0$

$$\phi_b=\phi(\eta_c)\Rightarrow \boxed{\phi_{\vec{k}}(\eta)=\phi_{b\vec{k}}\,\frac{1-ik\eta}{1-ik\eta_c}\,e^{ik(\eta-\eta_c)}}$$

If we take $\eta_c\to 0$, then the denominator $\to 1$.

(c) $S=-\frac{1}{2}\int\frac{d^3x\,d\eta}{\eta^2}\left[(\partial_\eta\phi)^2-(\partial_i\phi)^2\right]$

$$=\frac{1}{2}\int\frac{d\eta}{\eta^2}\int\frac{d^3k}{(2\pi)^3}\left[\partial_\eta\phi_{\vec{k}}\,\partial_\eta\phi_{-\vec{k}}-k^2\phi_{\vec{k}}\phi_{-\vec{k}}\right]$$

$$S_{cl}=S[\phi_{cl}]=-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\,\phi_{-\vec{k}}\,\frac{1}{\eta^2}\,\partial_\eta\phi_{\vec{k}}\Big|_{\eta=\eta_c}$$

$$\partial_\eta\phi_{\vec{k}}=c_1(-ik+ik+k^2\eta)e^{ik\eta}=c_1k^2\eta\,e^{ik\eta}$$

so $S_{cl}=-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\,\phi_{b,-\vec{k}}\,\frac{1-ik\eta}{1-ik\eta_c}\,e^{ik(\eta-\eta_c)}\,\frac{1}{\eta^2}\,\phi_{b,\vec{k}}\,\frac{1}{1-ik\eta_c}\,k^2\eta\,e^{ik(\eta-\eta_c)}\Big|_{\eta=\eta_c}$

$$=-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\,\phi_{b,-\vec{k}}\,\frac{k^2}{\eta_c}\,\phi_{b,\vec{k}}\,\frac{1}{1-ik\eta_c}$$

Expanding about $\eta_c\approx 0$, we get

$$\boxed{S_{cl}=\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\,\phi_{b,\vec{k}}\,\phi_{b,-\vec{k}}\left(\frac{k^2}{\eta_c}+ik^3\right)}$$

The first piece is divergent as $\eta_c\to 0$, but it's a pure phase in the wavefunction $e^{iS_{cl}}$; the second term is the nontrivial term.

(d) This (EAdS) is related to dS by $\eta=iz$ and $R_{dS}=iR_{AdS}$.

so $\boxed{\phi_{\vec{k}}(z)=\phi_{b\vec{k}}\,\frac{1+kz}{1+kz_c}\,e^{-k(z-z_c)}}$

$$+S_{EAdS}=+\frac{1}{2}\int\frac{d^3x\,dz}{z^2}\left[(\partial_z\phi)^2+(\partial_i\phi)^2\right]=i\,S_{dS}$$

(remember $\int_{z_c}^{\infty}dz=i\int_{-\infty}^{\eta_c}d\eta$

Group 1

So $$\boxed{e^{-S_{cl}^{EAdS}} = e^{\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\phi_{b,\vec{k}}\,\phi_{b,-\vec{k}}\left(-\frac{k^2}{z_c}+k^3\right)}}$$

(e) Back to the dS case,

$$\frac{\delta}{\delta\phi_b(\vec{k})}\frac{\delta}{\delta\phi_b(-\vec{k}')}e^{iS_{cl}}\Big|_{\phi_b=0} = \frac{\delta}{\delta\phi_{b,\vec{k}}}\frac{\delta}{\delta\phi_{b,-\vec{k}'}}e^{\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\phi_{b,\vec{k}}\phi_{b,-\vec{k}}\left(\frac{ik^2}{\eta_c}-k^3\right)}\Big|_{\phi_b=0}$$

$$= \boxed{(2\pi)^3\delta(\vec{k}-\vec{k}')\left(\frac{ik^2}{\eta_c}-k^3\right)}$$

This can be interpreted as the correlation function of a CFT, once the first term (divergent as $\eta_c\to 0$) is subtracted by a local counterterm in the CFT. Thinking about $e^{iS_{cl}} = Z_{CFT}$ is the conjecture of dS/CFT.

Going to position space, $\langle O_{\vec{k}}\, O_{-\vec{k}'}\rangle = -(2\pi)^3\delta(\vec{k}-\vec{k}')\,k^3$,

$$\langle O(\vec{x})O(\vec{y})\rangle = \int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left[-k^3\right] \sim \frac{1}{|\vec{x}-\vec{y}|^6}$$

so the conformal dimension of O dual of massless ϕ is 3.

There does not seem to be IR divergences in the CFT. The $\eta_c\to 0$ divergence is an IR divergence in the bulk, and a UV divergence in the CFT.

(d). $$\langle 0|T\phi_{\vec{k}}(\eta)\phi_{\vec{k}'}(\eta')|BD\rangle = \int D\phi\Big|_{\phi_b=0}\phi_{\vec{k}}(\eta)\,\phi_{\vec{k}'}(\eta')\,e^{iS_0}$$

In order to compute this, let's define a partition function

$$Z[J] = \int D\phi\Big|_{\phi_b=0} e^{iS_0 + i\int J\phi\sqrt{-g}}\ , \quad S_0 = -\frac{1}{2}\int\sqrt{-g}\,(\nabla\phi)^2$$

so the classical EOM becomes $\nabla^2\phi = -J$ with boundary condition $\phi(\eta_c) = \phi_b = 0$

~~Let's solve a Green's function~~

$$\cancel{\nabla^2 G(\vec{x},\vec{x}';\eta,\eta') = \delta(\vec{x}-\vec{x}')\delta(\eta-\eta')/\sqrt{-g}}$$

$$J_{\vec{k}} = \nabla^2\phi_{\vec{k}} = \eta^4\left[+\partial_\eta\left(\frac{1}{\eta^2}\partial_\eta\phi_{\vec{k}}\right)+k^2\frac{1}{\eta^2}\phi_{\vec{k}}\right]$$

Consider a Green function: $f_{\vec{k}} \equiv \frac{1}{\sqrt{2k^3}}(1-ik\eta)e^{ik\eta}$

$$\cancel{-\nabla^2 G_{\vec{k}}(\eta) = \delta(\eta-\eta')/\sqrt{-g}} \quad -\nabla^2 G_{\vec{k}}(\eta;\eta') = \delta(\eta-\eta')/\sqrt{-g}$$

or $\partial_\eta\left(\frac{1}{\eta^2}\partial_\eta G_{\vec{k}}(\eta;\eta')\right) + \frac{k^2}{\eta^2}G_{\vec{k}} \doteq \delta(\eta-\eta')$

so $G_{\vec{k}} = \begin{cases} c_1 f_k, & \eta<\eta' \\ c_2 f_k + c_3 f_k^* & \eta>\eta' \end{cases}$

We want ⓐ continuity at η': $c_1 f_k(\eta') = c_2 f_k(\eta') + c_3 f_k^*(\eta')$ ①

ⓑ discontinuity of first derivative:

$\frac{1}{\eta^2}\partial_\eta G_{\vec{k}}\Big|_{\eta'-\varepsilon}^{\eta'+\varepsilon} = 1 \Rightarrow c_2 f_k' + c_3 f_k'^* - c_1 f_k' = \eta'^2$ ②

ⓒ boundary condition at η_c: $c_2 f_k(\eta_c) + c_3 f_k^*(\eta_c) = 0$ ③

Take $\eta_c \to 0$, so $c_2 + c_3 = 0$

① $\Rightarrow \frac{c_1}{c_2} = \frac{f_k - f_k^*}{f_k}$ ② $\Rightarrow c_2(f_k' - f_k'^*) - c_1 f_k' = \eta'^2$

One can solve $c_1(\eta')$, $c_2(\eta')$. $c_2 \overset{(\eta')}{=} C_k(\eta')\, f_k(\eta')$

so $\phi_{\vec{k}} = \int d\eta'\, J_{\vec{k}}(\eta')\, G_{\vec{k}}(\eta,\eta')\sqrt{-g(\eta')}$ Define $\tilde{J} = \sqrt{-g}\,J$

$$S_0 + \int J\phi\sqrt{-g} = \frac{1}{2}\int J\phi\sqrt{-g} = \frac{1}{2}\int \tilde{J}\phi = \frac{1}{2}\int \frac{d\eta\, d\eta'\, d^3k}{(2\pi)^3} \times$$
$$\times\ \tilde{J}_{-\vec{k}}(\eta)\tilde{J}_{\vec{k}}(\eta')\, G_k(\eta;\eta')$$

so $Z[J] = e^{i\frac{1}{2}\int \frac{d\eta d\eta' d^3k}{(2\pi)^3}\tilde{J}_{-\vec{k}}(\eta)\tilde{J}_{\vec{k}}(\eta')G_k(\eta;\eta')}$

$$\langle 0|T\phi_{\vec{k}}(\eta)\phi_{\vec{k}'}(\eta')|BD\rangle = -\frac{\delta}{\delta\tilde{J}_{\vec{k}}(\eta)\,\delta J_{\vec{k}'}(\eta')} Z[J]\Big|_{J=0}$$

$$= -i(2\pi)^3\delta(\vec{k}+\vec{k}')\,G_k(\eta,\eta')$$

For $\eta>\eta'$ we have

$$\langle 0|T\phi_{\vec{k}}(\eta)\phi_{\vec{k}'}(\eta')|BD\rangle = \boxed{i(2\pi)^3\delta(\vec{k}+\vec{k}')\,C_k(\eta')f_k(\eta')\left[f_k^*(\eta) - f_k(\eta)\right]}$$

Group 1
Maldacena's Problem set 1.3

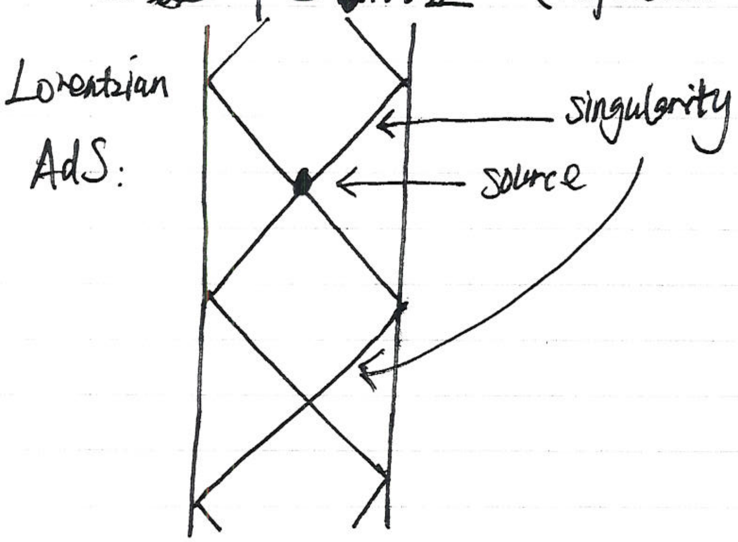
(f) It is much easier to solve the Laplace's equation directly, using the boundary conditions to pick out the correct position-space Green function, instead of summing the result of part (e) over k.

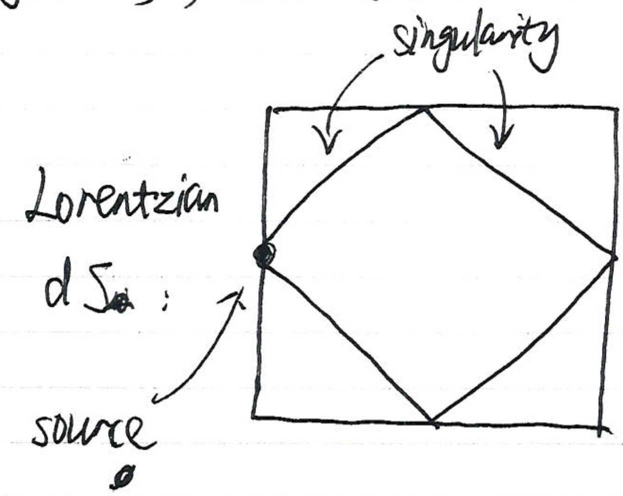
In EAdS4, the Green function has a singularity at the source and goes to zero at the boundary:

$$G(\rho) \sim \frac{1}{\cosh^6(\rho/2)}\, {}_2F_1\left[3, 2, 4, \frac{1}{\cosh^2(\rho/2)}\right]$$

where ρ is the geodesic distance. This is the Green's function we use in AdS/CFT.

In Lorentzian AdS4, we use $G(\rho)$ with ρ being the geodesic distance in Lorentzian AdS4 (timelike separation gives imaginary ρ). It is singular at $\rho=0$ (lightcone) or $\rho \in \pi i \mathbb{Z}$ (reflected lightcones), like this:





In dS4, the Greens function $\langle \phi_b=0 | T\phi(\eta, \vec{x})\phi(\eta', \vec{x}') | BD \rangle$ satisfies the same boundary condition — going to zero as $\eta \to 0$, so it must be given by the same $G(i\rho)$, with i appearing because $R_{AdS} \to iR_{dS}$ (remembering ρ is in terms of R_{dS})

This is <u>NOT</u> the Green function $\langle BD | T\phi\phi | BD \rangle$ which does not have singularity on the antipodal points. $G(i\rho)$ has singularity on both the lightcone ($\rho=0$) or the antipodal points ($\rho=\pi$). See above.

Group 1

Maldacena's Problem set 1.4 and 1.5

1.4 : $S_E = \frac{R_{AdS}^2}{16\pi G_N}\left[-\int_{\Sigma_4}\sqrt{g}(-12+6)\right]$

$= \frac{6R_{AdS}^2}{16\pi G_N}\int_0^{\rho_c}\sinh^3\rho\, d\rho\ \underbrace{Vol(S^3)}_{2\pi^2}$

$$\int_0^{\rho_c}\sinh^3\rho\, d\rho = \int_0^{\rho_c}\left(\frac{e^{\rho}-e^{-\rho}}{2}\right)^3 d\rho = \frac{1}{8}\int_0^{\rho_c} d\rho\left(e^{3\rho}-3e^{\rho}+3e^{-\rho}-e^{-3\rho}\right)$$

$$= \frac{1}{8}\left[\frac{1}{3}(e^{3\rho_c}-1)-3(e^{\rho_c}-1)-3(e^{-\rho_c}-1)+\frac{1}{3}(e^{-3\rho_c}-1)\right]$$

$$= \frac{1}{24}e^{3\rho_c}-\frac{3}{8}e^{\rho_c}+\underbrace{\left(-\frac{1}{24}+\frac{3}{8}+\frac{3}{8}-\frac{1}{24}\right)}_{3/4}+O(e^{-\rho_c})$$

Discarding first two divergent terms, we have

$$S_E = \frac{R_{AdS}^2}{16\pi G_N}(2\pi^2)\left(6\times\frac{3}{4}\right)$$

$$= \frac{9\pi R_{AdS}^2}{16 G_N} \qquad \boxed{\psi = Z \sim e^{-S_E} \sim e^{-\frac{9\pi R_{AdS}^2}{16 G_N}}}$$

1.5: Consider S^4 : $ds^2 = d\theta^2 + \cos^2\theta\, d\Omega_3^2$

with $S_E = \frac{R_{dS}^2}{16\pi G_N}\left[-\int_{\Sigma_4}\sqrt{g}(R-6)\right]$

where Σ_4 = hemisphere of S^4. $(-\frac{\pi}{2}\leq\theta\leq\theta_c)$

so $S_E = -\frac{6R_{dS}^2}{16\pi G_N}\int_{\Sigma_4}\sqrt{g} = -\frac{6R_{dS}^2}{16\pi G_N}\int_{-\frac{\pi}{2}}^{\theta_c} d\theta\cos^3\theta\ Vol(S^3)$

Since $\int_{-\frac{\pi}{2}}^{\theta_c} d\theta\cos^3\theta = \int_0^{\theta_c+\frac{\pi}{2}} d\theta\sin^3(\theta)$

so $S_E^{dS} = S_E^{AdS}\left[R_{AdS}\to iR_{dS},\ \rho_c\to i(\theta_c+\frac{\pi}{2})\right]$

Contine back to dS ; $\theta\to i\tau$ so $\tau_c = -i\theta_c = \rho_c + i\frac{\pi}{2}$

so $S_E^{dS} = -\frac{6R_{dS}^2}{16\pi G_N}(2\pi^2)\left(\frac{3}{4}\right) = -\frac{9\pi R_{dS}^2}{16 G_N}$, $\boxed{\psi \sim e^{\frac{9\pi R_{dS}^2}{16 G_N}}}$