1. Problem set 2: Integrability

Problem 1:

For the Sinh-Gordon theory, solve the asymptotic Bethe Ansatz equation in the two particle sector, for the case of zero total momentum. What is the energy of the lowest order solution ? How does it depend on m and L ? For $mL \gg 1$, do they behave like fermions? Why? What is the low energy scattering phase?. Can you compute the correction to the energy for an excited state with relatively large momentum ?. The S matrix is

$$S(\theta) = \frac{\sinh\theta - i\sin\pi p}{\sinh\theta + i\sin\pi p} \ , \qquad 0 < p \leq 1/2 \tag{1.1}$$

Does this S matrix have a bound state?. Compare to the scattering of similar massive particles in Sinh Gordon which gives something like (1.1) but with the opposite sign for p. For the sine gordon can you find the energy of the bound states ?.

Problem 2:

Show that a single massive scalar field in two dimensions with interaction ϕ^5 is not integrable. (Hint: does the $2 \to 3$ amplitude vanish ?)

Problem 3:

Consider a chain of spins. (Spin 1/2 representations of $SU(2)$). Consider the most general $SU(2)$ invariant nearest neighbor Hamiltonian. Show that, up to a constant shift, it is the Heisenberg one. Show that is is given by the permutation operator. Why is this?. Hint use group theory and the decomposition of two spin 1/2 representations into spin one and zero.

2. $\mathcal{N} = 4$ super Yang Mills spin chains

Problem 1:

Consider the operator $...ZZZZWZZZZ...$ with an W with some momentum. Understand the computation of the one loop anomalous dimension. Consider the diagrams that come from the term in the potential of the form $Tr\{[Z, W][\bar{Z}, \bar{W}]\}$. Understand the two terms that appear and their relative sign. Then compute the logarithmic terms in the correction to the two point function of operators $\sum_l e^{ipl} Tr[..ZZZZWZZZ...]$, where W is at position l, and its complex conjugate? Here we have been vague about the ... . You can consider a concrete operator such as $\sum_l Tr[WZ^l WZ^{J-l}]e^{ipl}$ and compute the anomalous dimension.

Problem 2:

Look up in Wess and Bagger the supersymmetry transformations of a theory with a superpotential. Conclude that $\delta_Q W \sim \psi$ and $\delta_Q \psi \sim [Z, W]$.

Problem 4:

Show that in a closed chain the sum of central charges κ_i vanishes, once we take into account the extra phase factors and momentum conservation $e^{i \sum_{i=1}^{n} p_i} = 1$. In other words the full generator is

$$\hat{K} \propto \sum_j f_j (1 - e^{ip_j}) \ , \qquad \text{where} \quad f_i = \prod_{k=1}^{j-1} e^{ip_k} \tag{2.1}$$

where the sum is over all the impurities in the chain.

Imagine that you assume that $\hat{K}$ has some expression of the form

$$\hat{K} \propto \sum_j f_j h(p_j) \ , \qquad \text{where} \quad f_i = \prod_{k=1}^{j-1} e^{ip_k} \tag{2.2}$$

with h some unknown function of p_j (and λ). The factor of f_j have to emerge since we know that the $\{QQ\}$ anticommutator inserts a Z. Now let's fix h as follows. We first make the further assumption that for small p $h(p)$ is linear in p. (Why is this a very reasonable assumption?). Then we will demand that on any closed chain with a set of particles conserving momentum we should have that $\hat{K}$ is zero. Choosing appropriate sets of particles show that h has the form in (2.1). (Hint: Consider a particle of momentum p_1 which is arbitrary, $0 < p_1 < 2\pi$. Then consider particles $p_2, \cdots, p_n$ with infinitesimal momentum, but that sums to $2\pi - p_1$, and take the large n limit. Use the fact that h is linear in p for small p and replace the sums by an integrals.)

This exercise shows that the functional form of h is fixed to (2.1) for all values of the coupling. Have we used integrability in this argument ?

Problem 4:

Consider the $SU(2|2)$ algebra with the extra central charges. The full algebra is given in (2.5) and (2.6) of hep-th/0511082 . Check that the fundamental representation has the energy stated in the lectures. That is shown in the same paper, but you can check it yourselves.

3. References

A great reference on integrability of 1+1 dimensional systems and the construction of the asymptotic S matrices for sine Gordon and the $O(n)$ models is:

Factorized s Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models. A. B. Zamolodchikov and Al. B. Zamolodchikov, Annals Phys.120:253-291,1979.

A nice reference on spin chains and their symmetries is:

Algebraic aspects of Bethe Ansatz , L.D. Faddeev, hep-th/9404013

A good place to start reading about the $N = 4$ Yang Mills spin chain is:

The $SU(2|2)$ *dynamic S-matrix* , N. Beisert, hep-th/0511082

Table 1 in hep-th/0511082 contains the full matrix form of the S matrix. The phase factor is given in an integral form in (46) (47) of hep-th/0703104. It was derived orginally in hep-th/0609044 and hep-th/0610251

The computation of the cusp anomalous dimension can be found in hep-th/0610251 (but you will need to have read hep-th/0603157).