1. Problem set 1: Introduction to the QFT/QG duality

Problem 1:

Consider the AdS metric given by the hyperboloid $Z^2 = -1$ in $R^{2,4}$. Write the metric in Poincare coordinates

$$x^\mu = \frac{Z^\mu}{Z^{-1} + Z^4}, \qquad z = \frac{1}{Z^{-1} + Z^4} \tag{1.1}$$

Write it also in "global coordinates"

$$Z^{-1} + iZ^0 = e^{i\tau} \cosh\rho \ , \qquad Z^i = \sinh\rho \, n^i \ , \qquad \vec{n}^2 = 1 \tag{1.2}$$

Get an understanding of the boundary of AdS and the Penrose diagram of AdS space.

Problem 2:

Find geodesics in Euclidean AdS, commonly known as hyperbolic space. What are they in the lorentzian AdS space.

Problem 3:

Consider a scalar field and solve the wave equation in AdS space in Poincare coordinates. You might want to do first the case of a massless field and then of a massive field. Solve the equation in Fourier space along the boundary directions. Understand the behavior of the two solutions near the boundary. Find the action for fixed boundary conditions at $z = \epsilon$. For the computation of the correlator you will need to impose that the solution decreases for large z. Understand the possible divergent terms. Distinguish between local and non-local contributions directly in Fourier space. (Hint: polynomial contributions in p^2 are local upon Fourier transformation). How would you do that? Consider the massless case too.

Problem 4:

Consider a black hole in AdS space with metric

$$ds^2 = R^2 \left[-dt^2 (1 - \frac{r_0^4}{r^4}) r^2 + r^2 d\vec{x}^2 + \frac{dr^2}{r^2 (1 - \frac{r_0^4}{r^4})} \right] \tag{1.3}$$

Can you compute the temperature?. (Hint: Send $t \to i\tau$ and set $\tau \sim \tau + \beta$, $\beta = 1/T$. Then demand that the space is non-singular at $r = r_0$.) Compute the entropy, or entropy per unit volume. Can you compute the mass density ? (Hint: compute the vev of the stress tensor by consider the traceless part of the metric). What is the specific heat ? Is it positive? (You can check that the specific heat is negative for Schwarzschild black hole in flat space.) Understand how scaling symmetry determines the dependence of the free energy (and the entropy) on the temperature.

Problem 5:

Consider the $AdS_5 \times S^5$ solution of type IIB supergravity. The relevant terms in the ten dimensional supergravity action have the form (up to numerical coefficients)

$$S = \int d^1 0x \sqrt{g}(e^{-2\phi}R + (\nabla\phi)^2 + F_5^2) \qquad F_5 = *_{10}F_5 \tag{1.4}$$

in string frame metric. (String frame means that the metric appearing in (1.4) is the one that appears in the string action as $S_{\rm string} \sim \frac{1}{2\pi\alpha'}(\text{Area}) \propto \int d^2x\sqrt{g_{ind}}$ where the area is measured with the metric g in (1.4).) Here $F_5 = F_{\mu_1\mu_2\mu_3\mu_4\mu_5}$ is a five index antisymmetric field strength. It is self dual. The quantization condition has the form

$$N \sim \int_{S^5} F_5 \tag{1.5}$$

and it does not involve the dilaton. Understand the scaling of the radius of curvature of AdS and the sphere with N and the string coupling. Understand the scaling with N of any supergravity computation. Understand the dependence on g and N of any computation that involves the action of a classical string worldsheet. What will be the dependence for Wilson loops. Or classical string solutions. What is the scaling for any computation involving D3 branes?

Problem 6:

Consider an AdS space with a hard wall cutoff at $r = r_0$ as a model for a confining theory. Consider a massless scalar field with Neuman boundary conditions at the wall. Find the spectrum of masses in the four dimensional theory. How do they scale with n for large n? Can you understand this scaling from a WKB approximation?. Now consider a string stretched along a spatial dimension. Can you compute its tension from the four dimensional point of view ?.

If you started with a massless graviton in the bulk and you got a massive spin two particle in four dimensions. Understand the counting of degrees of freedom.

Problem 7:

Show that a free field theory in ADS without gravity cannot be equal to a local quantum field theory on the boundary. Hint: Estimate the growth of the entropy in the bulk in global coordinates for high temperatures. Compare it to what you expect in a CFT on the boundary. (In the bulk just estimate the entropy in the central region with $\rho \sim 1.$). Once you include gravity, when does the above estimate fail ?. In other words, what gravitational effect makes the estimate of the free field theory invalid?. Does is make it make it consistent with CFT expectations?