Effective Field Theories

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Outline

- Introduction
- Reasons for using an EFT
- Dimensional Analysis and Power Counting
- Examples
- Loops
- Decoupling
- Field Redefintions and the Equations of Motion
- One-loop matching in a HQET example

Basic Idea

You can make quantitative predictions of observable phenomena without knowing everything.

The computations have some small (non-zero) error.

Can improve on the accuracy by adding a finite number of additional parameters, in a systematic way.

Key concept is locality — as a result one can factorize quantities into some short distance parameters (coefficients in the Lagrangian), and long distance operator matrix elements.

Examples

Chemistry and atomic physics depend on the interactions of atoms. The interaction Hamiltonian contains non-relativistic electrons and nuclei interacting via a Coulomb potential, plus electromagnetic radiation.

The only property of the nucleus we need is the electric charge Z.

The quark structure of the proton, weak interactions, GUTs, etc. are irrelevant.

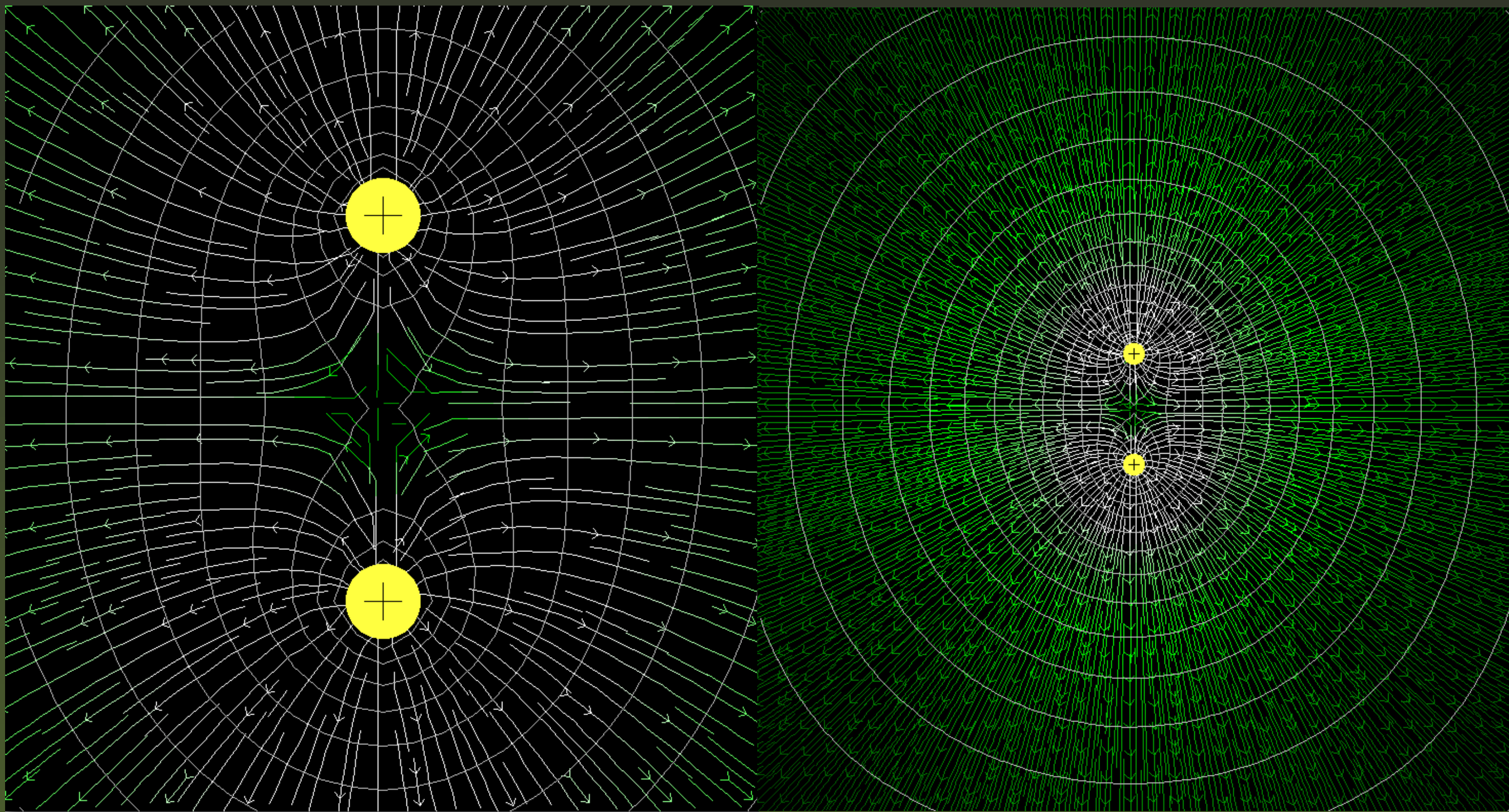
A more accurate calculation includes recoil corrections and needs m_p. The Hyperfine interaction needs μ_p

Charge radius, . . .

Weak interactions, . . .

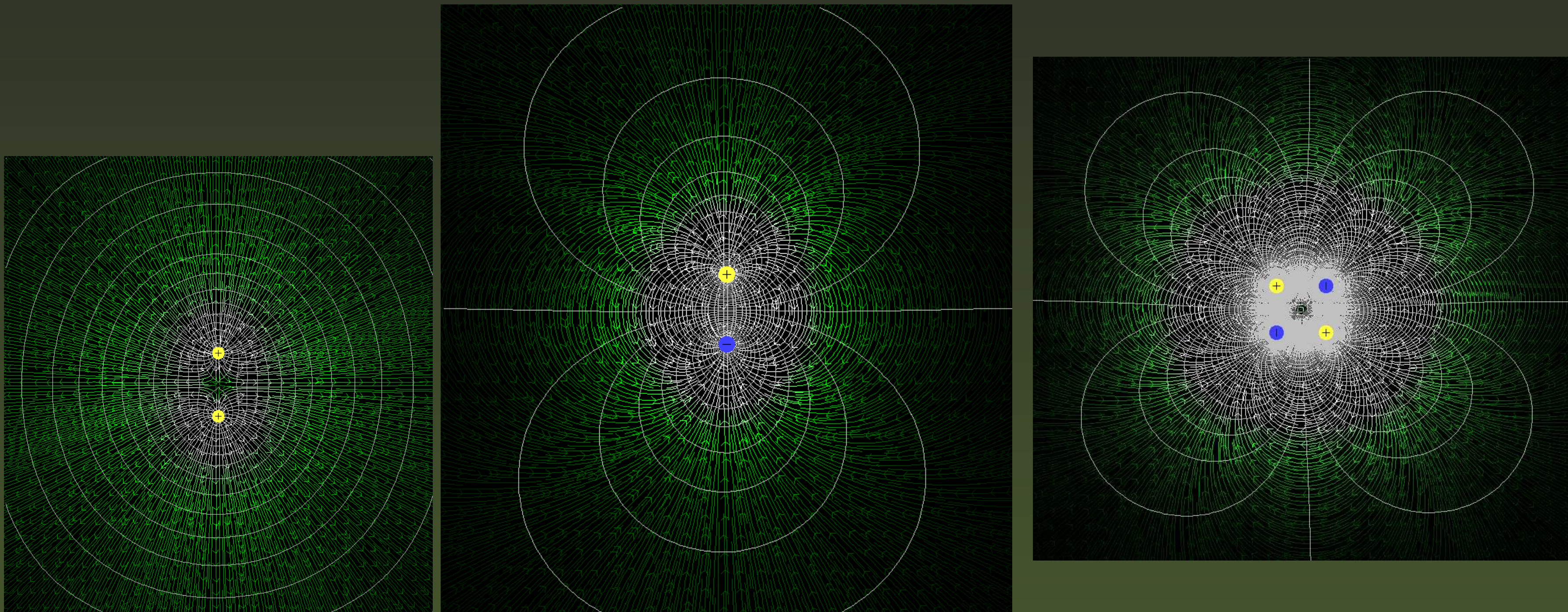
If one is interested in atomic parity violation, weak interactions are the leading contribution, and cannot be treated as a small correction.

Multipole Expansion



The field far away looks just like a point charge.

Multipole Expansion



Any collection of charges can be described at long distances by the multipole moment,

$$V \propto \frac{1}{r}\left(\frac{a}{r}\right)^{\ell}$$

Effective theory is a local quantum field theory with a finite number of low energy parameters.

There is a systematic expansion in a small parameters like a/r for the multipole expansion. [called power counting]

Keep as many terms as you need to reach the desired accuracy.

It is a quantum theory — one can compute radiative corrections (loops), renormalize the theory, etc. just as for QED or QCD.

EFT is the low-energy limit of a "full theory"

It is not a Lagrangian with form-factors $e \to eF(q^2/M^2)$

These are non-local, contain an infinite amount of information, and lead to a violation of power counting.

It is not just a series expansion of amplitudes in the full theory

$$F(q^2/M^2) \to F(0) + F'(0)\frac{q^2}{M^2} + \ldots$$

though it looks like this at tree-level.

The EFT is an interacting quantum theory in its own right.

One can compute using it without ever referring to the full theory from which it came.

The EFT has a different divergence structure from the full theory. The renormalization procedure is part of the definition of a field theory, not some irrelevant detail.

Examples of EFT

In some cases, one can compute the EFT from a more fundamental theory (typically, if it is weakly coupled).

- The Fermi theory of weak interactions is an expansion in p/M_W, and can be computed from the $SU(2) \times U(1)$ electroweak theory in powers of $1/M_W$, $\alpha_s(M_W)$, $\alpha(M_W)$ and $\sin^2\theta$.
- The heavy quark Lagrangian (HQET) can be computed in powers of $\alpha_s(m_Q)$ and $1/m_Q$ from QCD.
- NRQCD/NRQED
- SCET

Chiral perturbation theory: Describes the low energy interactions of mesons and baryons.

The full theory is QCD, but the relation between the two theories (and the degrees of freedom) is non-perturbative.

χPT has parameters that are fit to experiment. Has been enormously useful.

Standard model — don't know the more fundamental theory, and we all hope there is one.

Can use EFT ideas to parameterize new physics in terms of a few operators in studying, for example, precision electroweak measurements.

Dependence on high energy

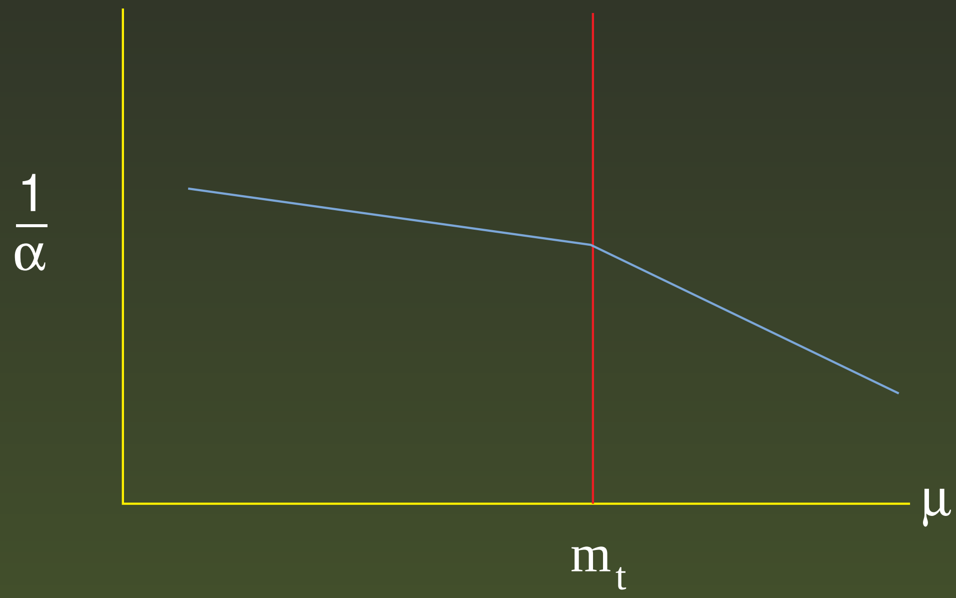
High energy dynamics irrelevant:

H energy levels do not depend on m_t — but this depends on what is held fixed as m_t is varied.

Usually, one takes low energy parameters such as m_p, m_e, α from low energy experiments, and then uses them in the Schrödinger equation.

But instead, hold high energy parameters such as $\alpha(\mu)$ and $\alpha_s(\mu)$ fixed at $\mu \gg m_t$.

$$m_t \frac{\mathrm{d}}{\mathrm{d}m_t}\left(\frac{1}{\alpha}\right) = -\frac{1}{3\pi}$$



The proton mass also depends on the top quark mass,

$$m_p \ \propto \ m_t^{2/27}$$

There are constraints from the symmetry of the high energy theory:

For example, the chiral lagrangian preserves C, P and CP because QCD does.

More interesting case: Non-relativistic quantum mechanics satisfies the spin-statistics theorem because of causality in QED.

Reasons for using EFT

- Every theory is an effective theory: Can compute in the standard model, even if there are new interactions at (not much) higher energies.
- Greatly simplifies the calculation by only including the relevant interactions: Gives an explicit power counting estimate for the interactions.
- Deal with only one scale at a time: For example the B meson decay rate depends on M_W, m_b and Λ_{QCD}, and one can get horribly complicated functions of the ratios of these scales. In an EFT, deal with only one scale at a time, so there are no functions, only constants.

- Makes symmetries manifest: QCD has spontaneously broken chiral symmetry, which is manifest in the chiral Lagrangian, and heavy quark spin-flavor symmetry which is manifest in HQET. These symmetries are only true for certain limits of QCD, and so are hidden in the QCD Lagrangian.
- Sum logs: Use renormalization group improved perturbation theory. The running of constants is not small, e.g. $\alpha_s(M_Z) \sim 0.118$ and $\alpha_s(m_b) \sim 0.22$. Fixed order perturbation theory breaks down. Sum logs of the ratios of scales (such as M_W/m_b).

- Efficient way to characterize new physics: Can include the effects of new physics in terms of higher dimension operators. All the information about the dynamics is encoded in the coefficients. [This also shows it is difficult to discover new physics using low-energy measurements.]
- Include non-perturbative effects: Can include Λ_{QCD}/m corrections in a systematic way through matrix elements of higher dimension operators. The perturbative corrections and power corrections are tied together. [Renormalons]

Dimensional Analysis

Effective Lagrangian (neglect topological terms)

$$L = \sum c_i O_i = \sum L_D$$

is a sum of local, gauge and Lorentz invariant operators.
The functional integral has

$$e^{iS}$$

so S is dimensionless.

Kinetic terms:

$$S = \int \mathrm{d}^d x \; \bar{\psi} \; i \not{D} \; \psi, \qquad S = \int \mathrm{d}^d x \; \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi$$

so

$$0 = -d + 2\,[\psi] + 1, \qquad 0 = -d + 2\,[\phi] + 2$$

Dimensions given by

$$[\phi] = (d{-}2)/2, \qquad [\psi] = (d{-}1)/2, \qquad [D] = 1, \qquad [g A_\mu] = 1$$

Field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \ldots$ so A_μ has the same dimension as a scalar field.

$$[g] = 1 - (d-2)/2 = (4-d)/2$$

In $d = 4$,

$$[\phi] = 1, \qquad [\psi] = 3/2, \qquad [A_\mu] = 1, \qquad [D] = 1, \qquad [g] = 0$$

Only Lorentz invariant renormalizable interactions (with $D \leq 4$) are

$$\begin{aligned}
D = 0: &\quad 1 \\
D = 1: &\quad \phi \\
D = 2: &\quad \phi^2 \\
D = 3: &\quad \phi^3, \bar{\psi}\psi \\
D = 4: &\quad \phi\bar{\psi}\psi, \phi^4
\end{aligned}$$

and kinetic terms which include gauge interactions.

Renormalizable interactions have coefficients with mass dimension ≥ 0.
In $d = 2$,

$$[\phi] = 0, \qquad [\psi] = 1/2, \qquad [A_\mu] = 0, \qquad [D] = 1, \qquad [g] = 1$$

so an arbitrary potential $V(\phi)$ is renormalizable. Also $\left(\bar{\psi}\psi\right)^2$ is renormalizable. In $d = 6$,

$$[\phi] = 2, \qquad [\psi] = 5/2, \qquad [A_\mu] = 2, \qquad [D] = 1, \qquad [g] = -1$$

Only allowed interaction is ϕ^3.

What Fields to use for EFT?

Not always obvious: Low energy QCD described in terms of meson fields.

NRQCD/NRQED and SCET: Naive guess does not work. Need multiple gluon fields.

Effective Lagrangian:

$$L_D = \frac{O_D}{M^{D-d}}$$

so in $d = 4$,

$$L_{\text{eft}} = L_{D \le 4} + \frac{O_5}{M} + \frac{O_6}{M^2} + \dots$$

An infinite number of terms (and parameters)

Power Counting

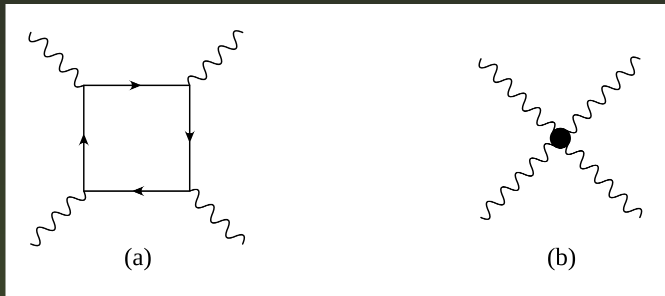
If one works at some typical momentum scale p, and neglects terms of dimension D and higher, then the error in the amplitudes is of order

$$\left(\frac{p}{M}\right)^{D-4}$$

A non-renormalizable theory is just as good as a renormalizable theory for computations, provided one is satisfied with a finite accuracy.

Usual renormalizable case given by taking $M \to \infty$.

Photon-Photon Scattering



$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha^2}{m_e^4}\left[c_1\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + c_2\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2\right].$$

(Terms with only three field strengths are forbidden by charge conjugation symmetry.)

e^4 from vertices, and $1/16\pi^2$ from the loop.

An explicit computation gives

$$c_1 = \frac{1}{90}, \qquad c_2 = \frac{7}{90}.$$

Scattering amplitude

$$A \sim \frac{\alpha^2 \omega^4}{m_e^4}$$

and

$$\sigma \sim \left(\frac{\alpha^2 \omega^4}{m_e^4}\right)^2 \frac{1}{\omega^2} \frac{1}{16\pi} \sim \frac{\alpha^4 \omega^6}{16\pi m_e^8} \times \frac{15568}{22275}$$

Proton Decay

The lowest dimension operator in the standard model which violates baryon number is dimension 6,

$$L \sim \frac{qqql}{M_G^2}$$

This gives the proton decay rate $p \to e^+\pi^0$ as

$$\Gamma \sim \frac{m_p^5}{16\pi M_G^4}$$

or

$$\tau \sim \left(\frac{M_G}{10^{15}\ \text{GeV}}\right)^4 \times 10^{30}\ \text{years}$$

Neutrino Masses

The lowest dimension operator in the standard model which gives a neutrino mass is dimension five,

$$\mathcal{L} \sim \frac{(HL)^2}{M_S}$$

This gives a Majorana neutrino mass of

$$m_\nu \sim \frac{v^2}{M_S}$$

or a seesaw scale of 6×10^{15} GeV for $m_\nu \sim 10^{-2}$ eV.

Absolute scale of masses not known. Only Δm^2 measured.

Rayleigh Scattering

$$L = \psi^\dagger \left(i\partial_t - \frac{p^2}{2M} \right) \psi + a_0^3 \ \psi^\dagger \psi \left(c_1 E^2 + c_2 B^2 \right)$$

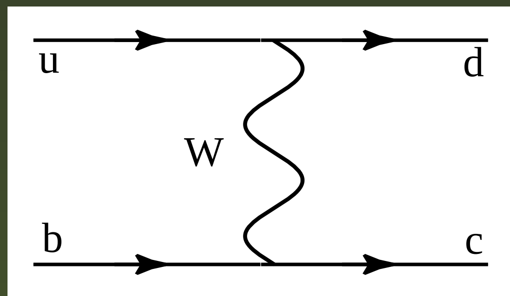
$$A \sim c_i a_0^3 \omega^2$$

$$\sigma \propto a_0^6 \ \omega^4 .$$

Scattering goes as the fourth power of the frequency, so blue light is scattered about 16 times mores strongly than red.

Low energy weak interactions

$$-\frac{ig}{\sqrt{2}}\; V_{ij}\;\bar{q}_i\,\gamma^\mu\,P_L\,q_j\,,$$



$$A = \left(\frac{ig}{\sqrt{2}}\right)^2 V_{cb}V_{ud}^*\;\;(\bar{c}\,\gamma^\mu\,P_L\,b)\,(\bar{d}\,\gamma^\nu\,P_L\,u)\left(\frac{-ig_{\mu\nu}}{p^2-M_W^2}\right),$$

$$\frac{1}{p^2 - M_W^2} = -\frac{1}{M_W^2}\left(1 + \frac{p^2}{M_W^2} + \frac{p^4}{M_W^4} + \dots\right),$$

and retaining only a finite number of terms.

$$A = \frac{i}{M_W^2}\left(\frac{ig}{\sqrt{2}}\right)^2 V_{cb} V_{ud}^* \;\left(\bar{c}\,\gamma^\mu\, P_L\, b\right)\left(\bar{d}\,\gamma_\mu\, P_L\, u\right) + \mathcal{O}\left(\frac{1}{M_W^4}\right).$$

$$L = -\frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* \;\left(\bar{c}\,\gamma^\mu\, P_L\, b\right)\left(\bar{d}\,\gamma_\mu\, P_L\, u\right) + \mathcal{O}\left(\frac{1}{M_W^4}\right),$$

$$\frac{G_F}{\sqrt{2}} \equiv \frac{g^2}{8M_W^2}.$$

Effective Lagrangian for μ decay

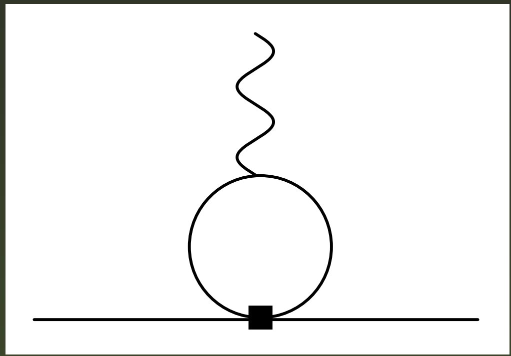
$$L = -\frac{4G_F}{\sqrt{2}} \; (\bar{e}\,\gamma^\mu\, P_L\,\nu_e)\,(\bar{\nu}_\mu\,\gamma^\mu\, P_L\,\mu) + \mathcal{O}\left(\frac{1}{M_W^4}\right),$$

Gives the standard result for the muon lifetime at lowest order,

$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3}.$$

The advantages of EFT show up in higher order calculations

Loops



Gives a contribution

$$\int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4} \frac{1}{k^2 - M_W^2} \frac{1}{k^2 - m^2} \sim \frac{1}{M_W^2} \int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4} \frac{1}{k^2 - m^2} \sim \frac{\Lambda^2}{M_W^2} \sim \mathcal{O}(1)$$

Similarly, a dimension eight operator has vertex k^2/M_W^4, and gives a contribution

$$I' \sim \frac{1}{M_W^4} \int d^4k \ \frac{k^2}{k^2 - m^2} \sim \frac{\Lambda^4}{M_W^4} \sim \mathcal{O}(1)$$

Would need to know the entire effective Lagrangian, since all terms are equally important. The reason for this breakdown is using a cutoff procedure with a dimensionful parameter Λ.

More generally, need to make sure that dimensionful parameters at the high scale do not occur in the numerator in Feynman diagrams.

In doing weak interactions, one should not have M_G or M_P appear in the numerator.

Need a renormalization scheme which maintains the power counting.

$\overline{\text{MS}}$

Need to use a mass independent subtraction scheme such as $\overline{\text{MS}}$: μ can only occur in logarithms, so

$$\frac{1}{M_W^2}\int d^4k\ \frac{1}{k^2-m^2} \sim \frac{m^2}{M_W^2}\log\frac{\mu^2}{m^2},$$

$$\frac{1}{M_W^4}\int d^4k\ \frac{k^2}{k^2-m^2} \sim \frac{m^4}{M_W^4}\log\frac{\mu^2}{m^2},$$

Expanding $1/(k^2-M_W^2)$ in a power series ensures that there is no pole for $k \sim M_W$, and so M_W cannot appear in the numerator. Dimensional regularization is like doing integrals using residues. Relevant scales given by poles of the denominator.

Expanding does not commute with loop integration

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \; \frac{1}{(k^2-m^2)(k^2-M^2)}$$

$$= \quad \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} + \log\frac{\mu^2}{M^2} + \frac{m^2\log(m^2/M^2)}{M^2-m^2} + 1\right]$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \; \frac{1}{(k^2-m^2)}\left[-\frac{1}{M^2} - \frac{k^2}{M^4} - \ldots\right]$$

$$= \quad \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon}\frac{m^2}{M^2-m^2} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{m^2}{M^2-m^2}\right]$$

Missing the non-analytic terms in M.

The $1/\epsilon$ terms do not have to agree, they are cancelled by counterterms which differ in the full and EFT. The two theories have different anomalous dimensions.

The term non-analytic in the IR scale, $\log(m^2)$ agrees in the two theories. This is the part which must be reproduced in the EFT. The analytic parts are local, and can be included as matching contributions to the Lagrangian.

The difference between the finite parts of the two results is

$$\begin{aligned}
&\frac{i}{16\pi^2}\left[\log\frac{\mu^2}{M^2}+\frac{m^2\log(\mu^2/M^2)}{M^2-m^2}+\frac{M^2}{M^2-m^2}\right]\\
=\;&\frac{i}{16\pi^2}\left[\left(\log\frac{\mu^2}{M^2}+1\right)+\frac{m^2}{M^2}\left(\log\frac{\mu^2}{M^2}+1\right)+\ldots\right]
\end{aligned}$$

The terms in parentheses are matching coefficients to a coefficient of order 1, order $1/M^2$, etc. They are analytic in m.

Note how $\log M/m \to \log M/\mu + \log \mu/m$, with the first part in the matching, and the second part in the EFT.

Power Counting Formula

Manifest power counting in p/M.

Loop graphs consistent with the power counting, since one can never get any M's in the numerator. If the vertices have $1/M^a$, $1/M^b$, etc. then any amplitude (including loops) will have

$$\frac{1}{M^a}\frac{1}{M^b}\ldots = \frac{1}{M^{a+b+\ldots}}$$

Correct dimensions due to factors of the low scale in the numerator, represented generically by p. (Could be a mass)

Only a finite number of terms to any given order.

Order $1/M$: L_5 at tree level

Order $1/M^2$: L_6 at tree level,
or loop graphs with two insertions of L_5.

General power counting result: you can count the powers of M. You can also count powers of p [Weinberg power counting formula for χPT]

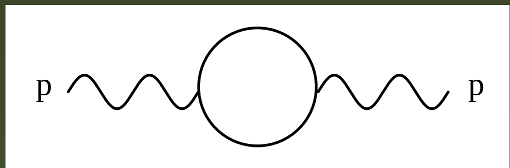
$$A \sim p^r, \qquad r = \sum_k n_k (k-4)$$

where n_k is the number of vertices of order p^k.

Decoupling

Heavy particles decouple from low energy physics. Obvious?

Not explicit in a mass independent scheme such as $\overline{MS}$.



$$i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\frac{1}{6\epsilon} - \int_0^1 dx\ x(1-x)\ \log\frac{m^2 - p^2 x(1-x)}{\mu^2}\right]$$

and we want to look at $p^2 \ll m^2$.

The graph is UV divergent.

Momentum Subtraction Scheme

Note that renormalization involves doing the integrals, and then performing a subtraction using some scheme to render the amplitudes finite.

Subtract the value of the graph at the Euclidean momentum point $p^2 = -M^2$ (the $1/\epsilon$ drops out)

$$-i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\int_0^1 dx\ x(1-x)\ \log\frac{m^2 - p^2 x(1-x)}{m^2 + M^2 x(1-x)}\right].$$

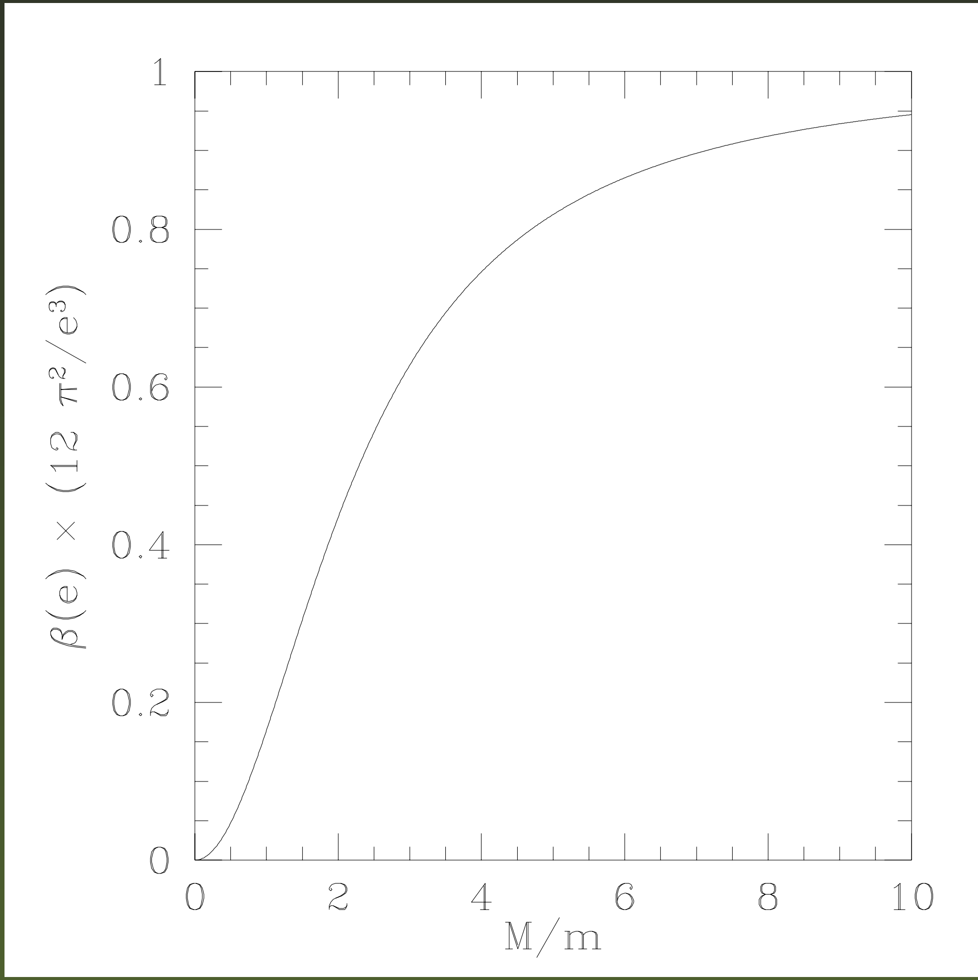
$$\begin{aligned}\beta\left(e\right) &= -\frac{e}{2}M\frac{\mathrm{d}}{\mathrm{d}M}\frac{e^2}{2\pi^2}\left[\int_0^1 dx\ x(1-x)\ \log\frac{m^2 - p^2 x(1-x)}{m^2 + M^2 x(1-x)}\right] \\ &= \frac{e^3}{2\pi^2}\int_0^1 dx\ x(1-x)\ \frac{M^2 x(1-x)}{m^2 + M^2 x(1-x)}.\end{aligned}$$

$m \ll M$ (light fermion):

$$\beta\left(e\right) \approx \frac{e^3}{2\pi^2}\int_0^1 dx\;\, x(1-x) = \frac{e^3}{12\pi^2}.$$

$M \ll m$ (heavy fermion):

$$\beta\left(e\right) \approx \frac{e^3}{2\pi^2}\int_0^1 dx\;\, x(1-x)\frac{M^2 x(1-x)}{m^2} = \frac{e^3}{60\pi^2}\frac{M^2}{m^2}.$$



cross-over

In the $\overline{\text{MS}}$ scheme:

$$-i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\int_0^1 dx\;\; x(1-x)\log\frac{m^2-p^2x(1-x)}{\mu^2}\right].$$

$$\begin{aligned}\beta\left(e\right) &= -\frac{e}{2}\mu\frac{d}{d\mu}\frac{e^2}{2\pi^2}\left[\int_0^1 dx\;\; x(1-x)\log\frac{m^2-p^2x(1-x)}{\mu^2}\right] \\ &= \frac{e^3}{2\pi^2}\int_0^1 dx\;\; x(1-x) = \frac{e^3}{12\pi^2},\end{aligned}$$

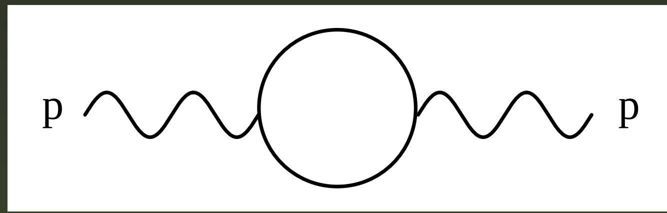
$$-i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\int_0^1 dx\ \ x(1-x)\log\frac{m^2}{\mu^2}\right],$$

Large logs cancel the wrong β-function contributions.

Explicitly integrate out heavy particles and go to an EFT.

Full theory: Includes fermion with mass m.

EFT: drop the heavy fermion (it no longer contributes to β)



Present in theory above m, but not in theory below m. Assume that $p \ll m$, so

$$\begin{aligned}
&\int_0^1 dx\ x(1-x)\ \log\frac{m^2 - p^2 x(1-x)}{\mu^2} \\
=\ &\int_0^1 dx\ x(1-x)\ \left[\log\frac{m^2}{\mu^2} + \frac{p^2 x(1-x)}{m^2} + \dots\right] \\
=\ &\frac{1}{6}\log\frac{m^2}{\mu^2} + \frac{p^2}{30m^2} + \dots
\end{aligned}$$

So in theory above m:

$$i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\frac{1}{6\epsilon} - \frac{1}{6}\log\frac{m^2}{\mu^2} - \frac{p^2}{30m^2} + \ldots\right] + c.t.$$

Counterterm cancels $1/\epsilon$ term (and also contributes to the β function).

$$i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[-\frac{1}{6}\log\frac{m^2}{\mu^2} - \frac{p^2}{30m^2} + \ldots\right]$$

The log term gives

$$Z = 1 - \frac{e^2}{12\pi^2}\log\frac{m^2}{\mu^2}$$

so that in the effective theory,

$$\frac{1}{e_L^2(\mu)} = \frac{1}{e_H^2(\mu)}\left[1 - \frac{e_H^2(\mu)}{12\pi^2}\log\frac{m^2}{\mu^2}\right]$$

One usually integrates out heavy fermions at $\mu = m$, so that (at one loop), the coupling constant has no matching correction.

The p^2 term gives the dimension six operator

$$-\frac{1}{4}\frac{e^2}{2\pi^2}\frac{1}{30m^2}F_{\mu\nu}\partial^2 F^{\mu\nu}$$

and so on.

Even if the structure of the graphs is the same in the full and effective theories, one still needs to compute the difference to compute possible matching corrections, because the integrals need not have the same value. (next example)

This difference is independent of IR physics, since both theories have the same IR behavior, so the matching corrections are IR finite.

Note that nothing discontinuous is happening to any physical quantity at m. We have changed our description of the theory from the full theory including m to an effective theory without m. By construction, the EFT gives the same amplitude as the full theory, so the amplitudes are continuous through m.

All m dependence in the effective theory is manifest through the explicit $1/m$ factors and through logarithmic dependence in the matching coefficients (in e_L).

HQET

A.V. Manohar and M.B. Wise: Heavy Quark Physics, Cambridge University Press (2000)

Will not discuss the theory or its applications in any detail.

Use it to discuss EFT at one-loop, because all the calculations can be found in any field theory textbook which discusses QED at one loop.

The EFT describes a heavy quark with mass m_Q interacting with gluons and light quarks with momentum $k \ll m_Q$

Expansion in $1/m_Q$

The full theory is QCD, the heavy quark part is

$$L = \bar{Q}\left(i\not{D} - m_Q\right)Q$$

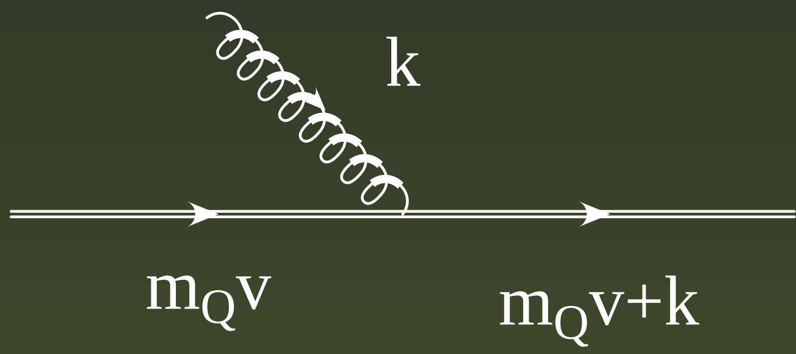
In the limit $m_Q \to \infty$, the heavy quark does not move when interacting with the light degrees of freedom. Even though for finite m_Q, the quark does recoil, the EFT is constructed as a formal expansion in powers of $1/m_Q$, expanding about the $m_Q \to \infty$ limit. Recoil effects are taken care of by $1/m_Q$ corrections.

Quark moving with fixed four-velocity v^μ

$$p = m_Q v^\mu + k \qquad k \ll m_Q$$

Quark Propagator

Look at the quark propagator:



$$i\frac{\not p + m_Q}{p^2 - m_Q^2 + i\epsilon}$$

$p = mv + k$, where k is called the residual momentum, and is of order Λ_{QCD}.

HQET Propagator

$$i\frac{m_Q \not{v} + \not{k} + m_Q}{(m_Q v + k)^2 - m_Q^2 + i\epsilon}$$

Expanding this in the limit $k \ll m_Q$ gives

$$i\frac{1+\not{v}}{2k\cdot v + i\epsilon} + \mathcal{O}\left(\frac{k}{m_Q}\right) = i\frac{P_+}{k\cdot v + i\epsilon} + \mathcal{O}\left(\frac{k}{m_Q}\right),$$

with a well defined limit.

$$P_+ \equiv \frac{1+\not{v}}{2}$$

Gluon Vertex

The quark-gluon vertex

$$-igT^a\gamma^\mu \to -igT^a v^\mu,$$

using the spinors and keeping the leading terms in $1/m_Q$.

In the rest frame: the coupling is purely that of an electric charge.

HQET $\mathcal{L}_0$

HQET Lagrangian:

$$\mathcal{L} \;=\; \bar{h}_v\,(x)\,(iD\cdot v)\,h_v\,(x)\,,$$

$h_v(x)$ is the quark field in the effective theory and satisfies

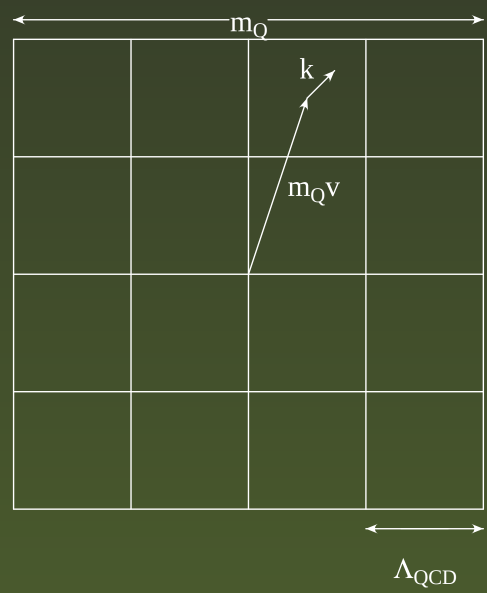
$$P_+ h_v\,(x) = h_v\,(x)\,.$$

h_v annihilates quarks with velocity v, but does not create antiquarks

Mainfest spin-flavor symmetry of $\mathcal{L}$

Dividing up momentum space

v appears explictly in the HQET Lagrangian.
h_v describes quarks with velocity v, and momenta within Λ_{QCD} of $m_Q v$.



quarks with velocity $v' \neq v$ are far away in the EFT.

EFT: look at only one box. Full: All of momentum space.

$1/m_Q$ Lagrangian

$$\mathcal{L} = \bar{h}_v \left(iv \cdot D\right) h_v + c_k \frac{1}{2m_Q} \bar{h}_v \left(iD_\perp\right)^2 h_v - c_F \frac{g}{4m_Q} \bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v$$

The $(iD_\perp)^2$ term violates flavor symmetry at order $1/m_Q$

$g\sigma_{\alpha\beta}G^{\alpha\beta}$ term violates spin and flavor symmetry at order $1/m_Q$

The coefficients c_k, c_F are fixed by matching, and are one at tree-level.

One can carry out the expansion to higher order in $1/m_Q$.

Field Redefinitions

Field redefinition

$$h_v \to \left[1 + \frac{1}{m_Q} X\right] h_v$$

changes the effective Lagrangian to

$$\begin{aligned}\mathcal{L} \;&=\; \bar{h}_v \left(iv \cdot D\right) h_v + \frac{1}{2m_Q} \bar{h}_v \left(iD_\perp\right)^2 h_v - \frac{g}{4m_Q} \bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v \\ &\qquad + \frac{1}{m_Q} h_v \left[\left(iv \cdot D\right) X + X^\dagger \left(iv \cdot D\right)\right] h_v + \mathcal{O}\left(\frac{1}{m_Q^2}\right)\end{aligned}$$

For $X = 1/2\,(iv \cdot D)$, one can replace

$$D_\perp^2 \to D_\perp^2 + \left(v \cdot D\right)^2 = D^2$$

Field redefinition change off-shell amplitudes, but not S-matrix elements. (Follows from the LSZ reduction formula)

The only thing that the effective theory and full theory have to agree on are S-matrix elements.

Effective Lagrangians which look different can reproduce the same S-matrix.

In a renormalizable field theory, the only field redefinition used is a rescaling $\psi \to Z^{1/2}\psi$ to keep the kinetic term properly normalized. Non-trivial field redefinitions would induce non-renormalizable terms with dimension > 4.

There is more freedom in an EFT, since we already have higher dimension terms.

Make field redefinitions consistent with the power counting.

Equations of Motion

Field redefinitions are related to using the equations of motion. The shift in L was proportional to $(v \cdot D)\psi$, which is the equation of motion from the leading order Lagrangian.

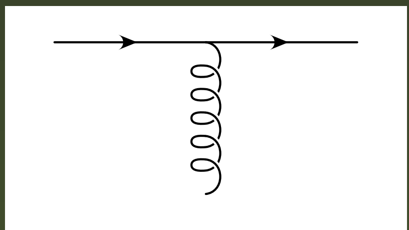
By making a transformation on a field, $\phi \to \phi + \xi f(\phi)$, one shifts the action by $\xi \delta S/\delta\phi + \mathcal{O}(\xi^2)$. So generically, one can eliminate terms proportional to the equation of motion by a field redefinition.

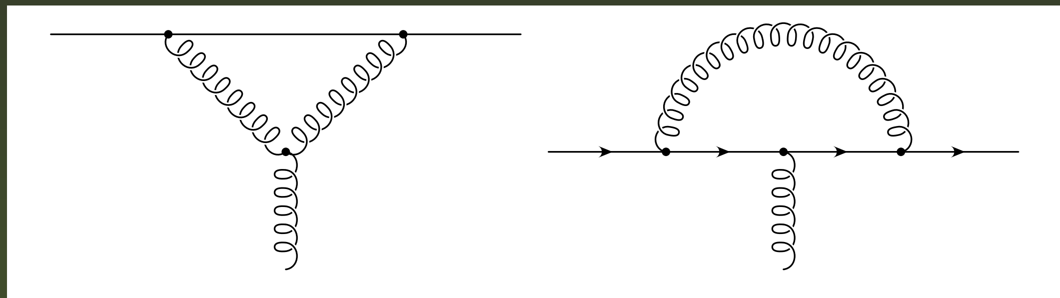
Note that one is using the classical equations of motion in the quantum theory (to all orders in $\hbar$).

One loop matching

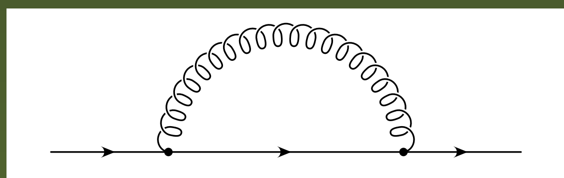
AM, PRD56 (1997) 230

Look at the gluon coupling to one-loop in the full theory:

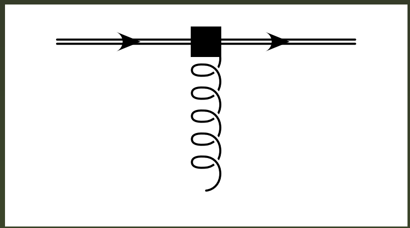


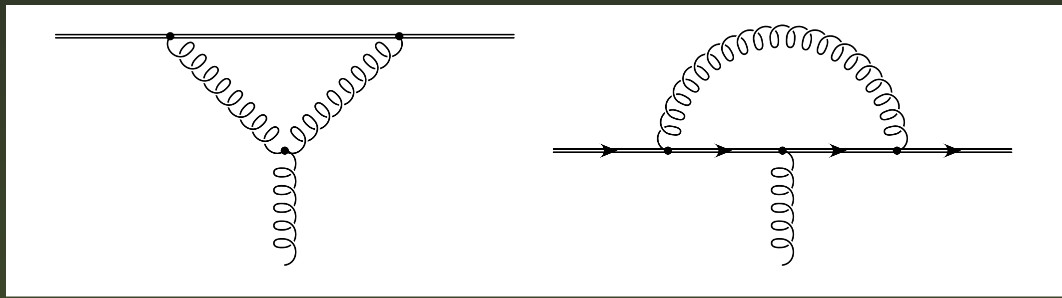


Note that we are matching an S-matrix element, the scattering of a quark off a low-momentum gluon, so the wavefunction graph must be included.

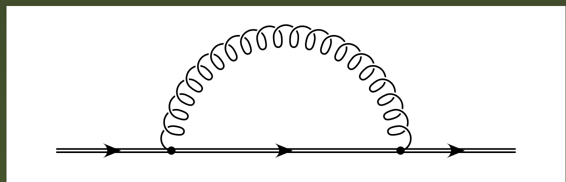


The effective theory graphs are:





and wavefunction renormalization



They look the same, but now

$$\gamma^\mu \to v^\mu, \qquad \frac{1}{\not{p} - m_Q} \to \frac{1}{v \cdot k}$$

The amplitude is

$$\Gamma^{(3)} = -igT^a \;\bar{u}\left(p'\right)\left[F_1\left(q^2\right)\gamma^\mu + iF_2\left(q^2\right)\frac{\sigma^{\mu\nu}q_\nu}{2m}\right]u\left(p\right),$$

by current conservation, so we are computing $F_{1,2}$ at one loop.

The graphs do not have this form unless you use background field gauge, which respects gauge invariance on the external gluon.

On-shell, the graphs are IR divergent.

What we want is the matching condition. It is given by the difference of the full and EFT amplitudes, and takes care of the fact that the two theories differ in the UV.

The EFT reproduces the IR behavior of the full theory, and so has the same IR divergences. Thus the matching corrections to the Lagrangian coefficients are IR finite, and well-defined.

One way to proceed: Assume the quarks are off-shell, $p^2 \neq m^2$ and compute full and EFT theory graphs.

$$p^2 - m_Q^2 \to (m_Q v + k)^2 - m_Q^2 \sim 2 m_Q v \cdot k$$

Matching using a IR Regulator

The graphs involve parameter integrals of the form

$$\int \frac{1}{m^2 z - q^2 x(1-x) z^2 - p_1^2 x z(1-z) - p_2^2 (1-x) z(1-z)}$$

Starts to look messy, because there are 4 scales in it.

The EFT graph is also messy. It involves q^2, $v \cdot k_1$ and $v \cdot k_2$ but not m.

Matching using Dim Reg for the IR

There is a much better procedure using dimensional regularization

The form factors are functions $F(q^2/m^2, \mu^2/m^2)$, and can have non-analytic terms such as $\log q^2$. These non-analytic terms arise from IR physics and so are the same in the full and EFT theory.

Compute $F(q^2/m^2, \mu^2/m^2, \epsilon)$ at finite ϵ and first expand in q^2/m^2 and then take the limit $\epsilon \to 0$. This drops all non-analytic terms in q, but we don't care since the effective Lagrangian is analytic in momentum.

$$x^\epsilon = x^\epsilon\Big|_{x=0} + \epsilon x^{\epsilon-1}\Big|_{x=0} + \ldots$$

In dim reg, all the terms are zero.
Then

$$F = F(0) + q^2 \frac{\mathrm{d}F(q^2)}{\mathrm{d}q^2}\Big|_{q^2=0} + \ldots$$

The derivatives of F are integrals of the form

$$F^{(n)}(0) \;\; = \;\; \int \mathrm{d}^4k \;\, f(k,m)$$

and depend on only a single scale.

$$\int \frac{1}{m^2 z - q^2 x(1-x) z^2 - p_1^2 x z(1-z) - p_2^2 (1-x) z(1-z)}$$

becomes

$$\begin{aligned} & \int \frac{1}{\left[\left(m^2 - q^2 x(1-x)\right) z^2\right]^{1+\epsilon}} \\ \rightarrow \quad & \int \frac{1}{\left[\left(m^2\right) z^2\right]^{1+\epsilon}}, \qquad\qquad (1+\epsilon) \int \frac{x(1-x) z^2}{\left[\left(m^2\right) z^2\right]^{2+\epsilon}} \end{aligned}$$

Full Theory Computation

$$F^{(n)}(0) \;=\; \frac{A_n}{\epsilon_{UV}} + \frac{B_n}{\epsilon_{IR}} + (A_n + B_n)\log\frac{\mu^2}{m^2} + D_n$$

UV divergences are cancelled by counterterms. IR divergences are real, and are an indication that you are not computing something sensible.

The derivatives of F in the EFT are integrals of the form

$$F^{(n)}(0) \;=\; \int \mathrm{d}^4 k \;\, f(k, v)$$

and are scaleless

EFT Computation

Scaleless integrals vanish in dim regularization.

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4} = \int \frac{d^d k}{(2\pi)^d} \left[\frac{1}{k^2 (k^2 - m^2)} - \frac{m^2}{k^4 (k^2 - m^2)} \right]$$

$$= \quad \frac{i}{16\pi^2} \left[\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right] = 0,$$

So the EFT form factor computation is

$$F^{(n)}(0) \quad = \quad \frac{E_n}{\epsilon_{UV}} - \frac{E_n}{\epsilon_{IR}} = 0$$

There can be no $\log$ terms and no constants.

Matching

Matching:

$$\text{full} = \text{eft} + c$$

which gives

$$\frac{A_n}{\epsilon_{UV}} + \frac{B_n}{\epsilon_{IR}} + (A_n + B_n)\log\frac{\mu^2}{m^2} + D_n + c.t. = \frac{E_n}{\epsilon_{UV}} - \frac{E_n}{\epsilon_{IR}} + c_n + c.t.$$

The ϵ_{UV} terms are cancelled by the counterterms, which are different in the two theories $(A_n \neq E_n)$

The IR divergence are the same, so $B_n = -E_n$.

$$c_n = (A_n + B_n)\log\frac{\mu^2}{m^2} + D_n$$

Summary

The anomalous dimension in the full theory is $-2A_n$

The anomalous dimension in the EFT is $-2E_n = 2B_n$

The IR divergence in the full theory is equal to the UV divergence in the EFT

The matching coefficient is given by the finite part of the full theory graph.

The coefficient of the $\log$ in the matching computation is the difference of the anomalous dimensions in the full and EFT theories.

Summing Logs

Suppose there is an infrared scale in the problem, λ^2. In the presence of this scale, the full theory IR divergence is cut off at λ^2, so the IR $\log$ term becomes

$$B \log \frac{\lambda^2}{m^2} = B \log \frac{\mu^2}{m^2} + B \log \frac{\lambda^2}{\mu^2}$$

This log of a ratio of scales is split into two logs, each involving a single scale. The first is in the matching at m, and the second is in the anomalous dimension in the EFT.

By matching at $\mu = m$, and then running from m to λ, one finds no large logs in the matching coefficient c.

The IR logs in the full theory are summed by the RGE in the effective theory.

Schematic Calculation

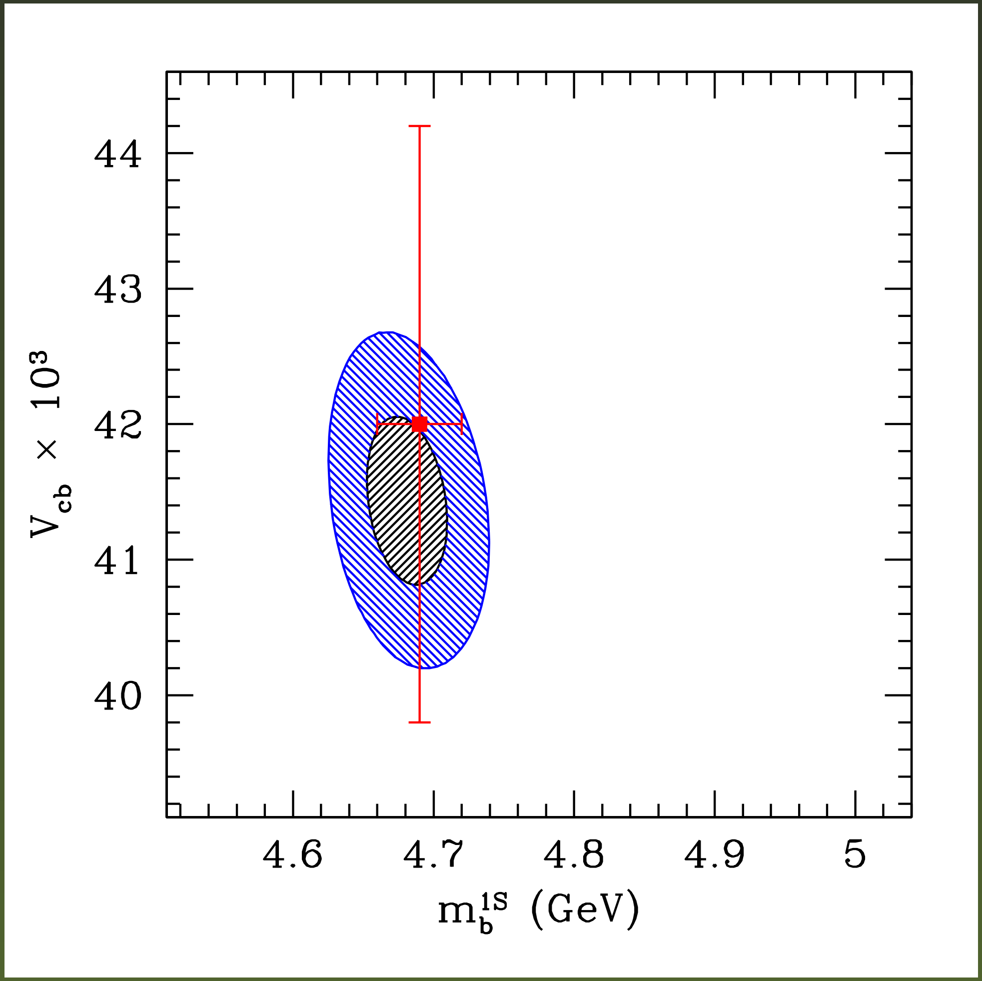
So how does one compute B decay?

1. Match onto the Fermi theory at M_W — $\alpha_s(M_W)$
2. Run the 4-quark operators to m_b — $(\alpha_s \log M_W/m_b)^n$.
3. Match onto HQET at m_b — $\alpha_s(m_b)$
4. Run in HQET to some low scale $\mu \sim \Lambda_{\rm QCD}$ — $(\alpha_s \log m_b/\Lambda_{\rm QCD})^n$
5. Evaluate non-perturbative matrix elements of operators in L — $\Lambda_{\rm QCD}/m_b$.

- One computation has been broken up into several much simpler calculations, each of which involves a single scale.
- Can sum the logarithms using RG improved perturbation theory, rather fixed order perturbation theory (which often breaks down)
- Both short distance and long distance corrections can be included in a systematic way to arbitrary accuracy (assuming you work hard enough).
- It is just the full theory computation, so there is no model dependence.

Application to B Decays

Bauer et al. PRD 70 (2004) 094017



$$m_b^{1S} = 4.68 \pm 0.03 \text{ GeV}, \qquad V_{cb} = (41.4 \pm 0.7) \times 10^{-3}$$