Introduction to the Standard Model, Set 2

1. Consider a generalization of the $SU(2)\times U(1)$ model involving k multiplets $\phi_i, i = 1\cdots k$, of complex scalars. The dimension of the i^{th} multiplet is $2t_i + 1$, where t_i can be $0, 1/2, 1, 3/2, \cdots$, and the elements have T^3 eigenvalues $t_i^3 = -t_i, -t_i + 1, \cdots, t_i$ (cf., the rotation group). Also, the i^{th} multiplet has weak hypercharge y^i. Assume that each multiplet has one electrically neutral component ϕ_i^0, i.e., with $q_i = t_i^3 + y_i = 0$, and that that component acquires a vacuum expectation value $\langle\phi_i^0\rangle = \nu_i/\sqrt{2}$.

   (a) Show that the mass eigenstates $W^\pm$, Z, and A are the same as in the standard model.

   (b) Calculate the W and Z masses in terms of g, g', t_i, t_i^3, and ν_i.

   (c) The ρ parameter, $\rho \equiv M_W^2/(M_Z^2\cos^2\theta_W)$ is predicted to be unity at the tree level in the standard model and in extensions involving additional Higgs doublets. Show that in the more general case

$$\rho = \frac{M_W^2}{M_Z^2\cos^2\theta_W} = \frac{\sum_{i=1}^k [t_i(t_i+1) - (t_i^3)^2]|\nu_i|^2}{2\sum_{i=1}^k (t_i^3)^2|\nu_i|^2}$$

   (d) Specialize to the case of one doublet and two triplets

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \Phi^{++} \\ \Phi^+ \\ \Phi^0 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix},$$

   where $\nu_\phi \equiv \sqrt{2}\langle\phi^0\rangle \gg \nu_\Phi \equiv \sqrt{2}\langle\Phi^0\rangle$ and $\nu_\phi \gg \nu_\Sigma \equiv \sqrt{2}\langle\Sigma^0\rangle$. Calculate ρ to leading nontrivial order in ν_Φ/ν_ϕ and ν_Σ/ν_ϕ.

   (e) Now consider the case of multiple Higgs doublets but no higher dimensional representations. Argue that the couplings of neutral physical Higgs bosons to fermions will no longer be flavor-diagonal. (Do not attempt to write the Higgs potential or find the exact Higgs mass eigenstates.)

2. The charged pion π^- decays approximately 99.99% of the time into $\mu^-\bar{\nu}_\mu$ and about 0.01% into $e^-\bar{\nu}_e$. The matrix element in the Fermi theory is

$$M = -i\frac{G_F}{\sqrt{2}}\ \bar{u}_\mu \gamma^\mu (1-\gamma^5) v_{\bar{\nu}_\mu}\ \langle 0|J_\mu^{h\dagger}|\pi^-(q)\rangle,$$

where the relevant part of the hadronic weak current is

$$J_\mu^{h\dagger} = \bar{u}\gamma_\mu(1-\gamma^5)\ d\ \cos\theta_c.$$

The hadronic matrix element $\langle 0|J_\mu^{h\dagger}|\pi^-(q)\rangle$ is a strong interaction quantity, which cannot be calculated in perturbation theory. However, its form can be derived using the symmetries of the *strong* interactions (weak effects in the matrix element are higher order in G_F).

(a) Use the parity conservation of the strong interactions to show that only the axial (γ^5) contribution is nonzero. (Note that the pion is a pseudoscalar, $P|\pi(\vec{q})\rangle = -|\pi(-\vec{q})\rangle$, where P is the space reflection operator).

(b) Use Lorentz invariance to argue that the matrix element must be of the form

$$\langle 0|J_\mu^{h\dagger}|\pi^-(q)\rangle = -i\cos\theta_c f_\pi\ q_\mu,$$

where f_π is a constant (the factor $-i\cos\theta_c$ is extracted for convenience). ($f_\pi \sim 132$ MeV, known as the pion decay constant, must be taken from experiment or calculated in lattice QCD. It is related to the "wave function" of the pion and to the spontaneous breaking of a global chiral symmetry of QCD.)

(c) Show that the decay rate is

$$\Gamma(\pi^-\to\mu^-\bar{\nu}_\mu) = \frac{G_F^2\cos^2\theta_c}{8\pi}\ f_\pi^2\ m_\mu^2 m_\pi\ \left(1-\frac{m_\mu^2}{m_\pi^2}\right)$$

(d) The rate for $\pi^-\to e^-\bar{\nu}_e$ is the same, except $m_\mu\to m_e$, explaining the 10^{-4} suppression. Interpret the suppression factor of m_e^2 in terms of angular momentum.