Introduction to the Standard Model

1. This problem concerns the gauge kinetic energy term for a non-abelian gauge symmetry.
Under a gauge transformation, the gauges fields $A^i_\mu, i = 1 \cdots N$, transform as

$$\vec{A}_\mu \cdot \vec{L} \quad \rightarrow \quad \vec{A}'_\mu \cdot \vec{L} \equiv U \vec{A}_\mu \cdot \vec{L} U^{-1} + \frac{i}{g} \left(\partial_\mu U \right) U^{-1}.$$

The $L^i, i = 1 \cdots N$ are a set of $n \times n$ dimensional representation matrices, satisfying

$$[L^i, L^j] = i c_{ijk} L^k,$$

where the c_{ijk} are the structure constants, and

$$\mathrm{Tr}(L^i L^j) = T(L) \delta^{ij},$$

where $T(L)$ is a constant that depends on the representation. (Note that any non-trivial irreducible representation can be used). g is the gauge coupling. U is the $n \times n$ matrix representation of the group element,

$$U \equiv e^{i \vec{\beta}(x) \cdot \vec{L}}.$$

(a) Prove that for small $|\beta^i|$,

$$A^{i\prime}_\mu = A^i_\mu - c_{ijk} \beta^j A^k_\mu - \frac{1}{g} \partial_\mu \beta^i + O(\beta^2)$$

(b) Prove that the kinetic Lagrangian

$$L_F = -\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu},$$

where

$$F^i_{\mu\nu} \equiv \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g c_{ijk} A^j_\mu A^k_\nu,$$

is gauge invariant for all $\vec{\beta}$ (i.e., *not* just small $|\beta^i|$).
Hint: Use

$$L_F = -\frac{1}{4T(L)} \mathrm{Tr}(\vec{F}_{\mu\nu} \cdot \vec{L})^2,$$

where

$$\vec{F}_{\mu\nu} \cdot \vec{L} = \partial_\mu \vec{A}_\nu \cdot \vec{L} - \partial_\nu \vec{A}_\mu \cdot \vec{L} + ig[\vec{A}_\mu \cdot \vec{L}, \vec{A}_\nu \cdot \vec{L}].$$

2. This problem involves a chiral fermion and complex scalar, with an internal global symmetry that may be both spontaneously and explicitly broken.

Consider the Lagrangian density

$$\begin{aligned} L &= \bar{\psi} i \not{\partial} \psi + \partial_\mu \phi^\dagger \partial^\mu \phi - h[\bar{\psi}_L \phi \psi_R + \bar{\psi}_R \phi^\dagger \psi_L] \\ &\quad - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 + a(\phi + \phi^\dagger), \end{aligned}$$

where $\psi = \psi_L + \psi_R$ is a fermion, ϕ is a complex scalar, $\lambda > 0$, $h > 0$, and $a \geq 0$.

(a) Show that L has a global chiral symmetry $U(1) \times U(1)$ for the case $a = 0$.

(b) Calculate the spectrum of the model (i.e., the masses) for the cases (i) ($\mu^2 > 0, a = 0$) and (ii) ($\mu^2 < 0, a = 0$).

(c) Calculate the spectrum for the cases (i) ($\mu^2 > 0, a > 0$) and (ii) ($\mu^2 < 0, a > 0$). In each case assume that $\sqrt{\lambda} a / |\mu|^3 \ll 1$, and keep only the leading nonzero term in that parameter.

(d) Interpret the spectrum in each of the above cases in terms of the symmetries and symmetry breaking.