SM (Collider) HW II

1. As discussed in the Lecture the parton distributions do not scale as in the naïve parton model but rather are expected to exhibit the scaling violation predicted by QCD. The structure of the expected renormalization of the parton distribution functions is summarized in terms of the DGLAP (also often called the "Altarelli-Parisi", *i.e*., the AP in DGLAP) splitting functions. As noted in Lecture 2 the lowest order expressions for these functions are given by

$$P_{qq}^{(0)}(x) = C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right],$$

$$P_{qg}^{(0)}(x) = T_R\left[x^2 + (1-x)^2\right],$$

$$P_{gq}^{(0)}(x) = C_F\left[\frac{1+(1-x)^2}{x}\right],$$

$$P_{gg}^{(0)}(x) = 2\,C_A\left[\frac{x}{(1-x)_+} + \frac{(1-x)}{x} + x(1-x)\right] + \delta(1-x)\frac{11C_A - 4n_f T_R}{6},$$

where the "+" notation means

$$\int_0^1 dx \frac{f(x)}{(1-x)_+} \equiv \int_0^1 dx \frac{f(x) - f(1)}{(1-x)}.$$

a) Verify that the corresponding anomalous dimensions (the moments of these functions, $\gamma(j) \equiv \int_0^1 dx x^{j-1} P(x)$) have the forms

$$\gamma_{qq}^{(0)}(j) = C_F\left[-\frac{1}{2} + \frac{1}{j(j+1)} - 2\sum_{k=2}^{j}\frac{1}{k}\right],$$

$$\gamma_{qg}^{(0)}(j) = T_R\left[\frac{2+j+j^2}{j(j+1)(j+2)}\right],$$

$$\gamma_{gq}^{(0)}(j) = C_F\left[\frac{2+j+j^2}{j\left(j^2-1\right)}\right],$$

$$\gamma_{gg}^{(0)}(j) = 2C_A\left[-\frac{1}{12}+\frac{1}{j(j-1)}+\frac{1}{(j+1)(j+2)}-\sum_{k=2}^{j}\frac{1}{k}\right]-\left(\frac{2}{3}\right)n_f T_R.$$

Now consider the evolution of the singlet quark distribution given by the sum

$$\Sigma(x) = \sum_i q_i(x)+\overline{q}_i(x),$$

which mixes with the gluon distribution via the evolution equation. In terms of the moments with evolution variable $t=\ln\left(Q^2/\Lambda_{QCD}^2\right)$ we have

$$\frac{d}{dt}\begin{pmatrix}\Sigma(j)\\ g(j)\end{pmatrix}=\frac{\alpha_s(t)}{2\pi}\begin{pmatrix}\gamma_{qq}(j) & 2n_f\gamma_{qg}(j)\\ \gamma_{gq}(j) & \gamma_{gg}(j)\end{pmatrix}\begin{pmatrix}\Sigma(j)\\ g(j)\end{pmatrix}.$$

b) Verify that for j = 2 there are two eigenvalues to the above evolution equation and that the corresponding anomalous dimensions are $\lambda_+=0$ (momentum conservation) and $\lambda_-=-\left(16/9+n_f/3\right)$ corresponding to the eigenvectors $\Sigma(2)+g(2)$ and $\Sigma(2)-3n_f g(2)/16$, respectively.

c) Use the result of b) to find the momentum fraction in quarks and that in gluons at truly asymptotic values Q^2.

2. The evolution of the distribution functions tends to build up the gluon distribution at small *x*, which will be important at the LHC. Here we consider this point in more detail. In the limit of small *x* and very large Q^2 the DGLAP equation is dominated by the small argument behavior of the splitting function P_{gg}.

a) Verify that in this limit the gluon distribution $G(x,t) = xg(x,t)$ satisfies the equation (again $t=\ln\left(Q^2/\Lambda_{QCD}^2\right)$)

$$\frac{dG(x,t)}{dt} \simeq \frac{3\alpha_s(t)}{\pi}\int_x^1 \frac{dy}{y} G(y,t).$$

b) Now use the 1-loop form for α_s and change variables to $\tau = \ln t$ and $\varsigma = (24/b_0)\ln(1/x)$ to show that the approximate equation we want to solve is

$$\frac{d^2 G(\varsigma,\tau)}{d\varsigma d\tau} \simeq \frac{1}{2} G(\varsigma,\tau).$$

c) Verify that at truly large values of both ζ and τ this equation is solved by

$$G(\varsigma,\tau) \sim e^{\sqrt{2\varsigma\tau}}$$

or

$$g(x,t) \sim \frac{1}{x}\exp\sqrt{\frac{48}{b_0}\ln\left(\frac{t}{t_0}\right)\ln\left(\frac{1}{x}\right)} \times \left(xg(x,t_0)\right).$$

d) To get a feeling for the size of this enhancement assume that the gluon distribution at Q_0 = 5 GeV is given by the following (fictitious) expression,

$$g(x) = \frac{420}{99}\frac{(1-x)^7}{x}.$$

Note that it already has the $1/x$ behavior at small x. Now evaluate the above enhancement factor for Q = 100 GeV at x = 0.01 with Λ_{QCD} = 0.1 GeV. How much larger is the evolved distribution at this x value, assuming that the above expression is relevant in the specified kinematic regime? (Take n_f = 5 for this estimate.)

3. Let us focus briefly on the Drell-Yan process, the production of a virtual photon in hadron-hadron collisions via the annihilation of a quark and antiquark (here ignore the possibility of Z production). The short distance ("hard") process is the time

reversed version of the electron-positron annihilation process. Thus $e^+e^- \to q\overline{q}$ becomes $q\overline{q} \to \mu^+\mu^-$ (or $q\overline{q} \to e^+e^-$), but where the specific choice of the muon pair typically rises from the desire to employ a lepton pair that is "easily" detected. This cross section must then be convoluted with the appropriate parton distribution functions. In terms of the scaled virtual photon mass $\tau = Q^2/s$ and the photon rapidity $y = \frac{1}{2}\ln\left[(q_0 - q_z)/(q_0 + q_z)\right]$, the "scaling" or parton model version of the cross section looks like

$$s\frac{d\sigma}{d\sqrt{\tau}dy} = \frac{8\pi\alpha^2}{3\sqrt{\tau}} g\left(\sqrt{\tau}e^y, \sqrt{\tau}e^{-y}\right),$$

where the (LO) parton "luminosity" function has the form

$$g\left(x_a, x_b\right) = \frac{1}{3}\sum_f e_f^2 \left\{q_f^a\left(x_a\right)\overline{q}_f^b\left(x_b\right) + \overline{q}_f^a\left(x_a\right)q_f^b\left(x_b\right)\right\}.$$

The label *a*,*b* correspond to the 2 incident hadrons. The explicit factor of 1/3 is required because the conventional normalization of the *pdf*s ($q, \overline{q}$) includes an implicit sum over colors. Here the quark-antiquark pair that annihilates must be of the same color. Thus the annihilation occurs for only 1/3 of the possible pairs.

Contrary to the collider physics we have focused on in the Lectures, consider now the case of pion beams incident on a nuclear target, *i.e.*, composed of the canonical nucleon $N = (p+n)/2$. If we focus on large $\tau = Q^2/s$ so that we can safely assume that the interaction is dominated by the valence quarks (and antiquarks), determine the expected (and observed) value of the ratio

$$R_{DY} \equiv \frac{\sigma\left(\pi^+ N \to \mu^+\mu^- X\right)}{\sigma\left(\pi^- N \to \mu^+\mu^- X\right)}.$$

4. In this exercise we want to become familiar with various features of collider kinematics. As noted in the Lecture the "real" rapidity and the pseudo-rapidity are defined by

$$\text{rapidity } = y_J = 0.5\ln\left(\frac{E+p_z}{E-p_z}\right)$$

$$\text{pseudo-rapidity } = \eta = -\ln\left(\tan\left(\frac{\theta}{2}\right)\right)$$

where the z direction is the direction of the beam.

a) Verify, as stated in the Lectures, that for any particle of mass M we can write

$$E = \sqrt{M^2 + p_T^2}\cosh y,\ p_z = \sqrt{M^2 + p_T^2}\sinh y,$$
$$p_T^2 = p_x^2 + p_y^2.$$

b) Prove that $\tanh\eta = \cos\theta$ and thus that η is easy to measure.

c) If particles are produced uniformly in longitudinal phase space with a differential distribution that looks like

$$dN = C\frac{dp_z}{E},$$

with C a constant, find the corresponding distribution in y, dN/dy.

d) Prove that the rapidity equals the pseudo rapidity, $\eta = y$, for a massless particle (and thus approximately for a relativistic particle, E ≫ M).

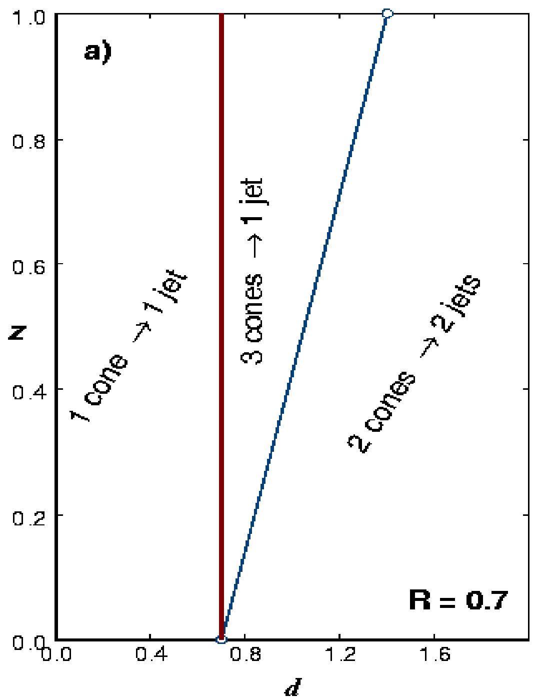
e) Prove that for a Lorentz transformation (boost) in the beam (z) direction, the rapidity, y, of every particle is shifted by a constant y_0, which is related to the boost velocity. Recall the form of such a boost to a reference frame moving in the z direction with velocity u (with respect to the original frame and with c = 1)

$$E' = \gamma\left(E - \beta p_z\right),$$
$$p'_z = \gamma\left(p_z - \beta E\right),$$
$$p'_x = p_x,\ p'_y = p_y,$$
$$\beta = u,\ \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

f) Consider a *Z* boson that is produced on-shell at the LHC in a $q\bar{q}$ annihilation process. The velocity of the *Z* boson is along the beam direction. What are the conditions relating x_1 and x_2, the momentum fractions of the quark and anti-quark? (Compare to the expressions in the previous exercise.)

5. Demonstrate that both the cone algorithm (without seeds) and the k_T algorithm are IRS at NLO in pQCD, *i.e.*, show that the "found" jet will have the same properties whether it contains a single parton or a pair of collinear partons with the same total momentum. Also show that the jet is unchanged by the emission of a (vanishingly) soft gluon. This does not require a slick argument. The idea is just to give you the opportunity to think through what it takes to be IRS.

6. Use the Snowmass definition of the iterative cone algorithm (i.e., E_T weighting instead of 4-vector addition) to show that the 2-parton phase space splits up as indicated in the figure in the Lecture. While this is really a 2-D problem in (y,ϕ), the fact that there are only 2 partons, which effectively lie in a plane, means we can think of it as a 1-D problem, *i.e.*, just the separation *d* in that plane.


a)
R = 0.7
1 cone → 1 jet
3 cones → 1 jet
2 cones → 2 jets
z
d
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.4
0.8
1.2
1.6