Two Component Spinors

The massless Dirac equation simplifies dramatically in the case that the fermion mass is zero. The equation

$$\not{D}\psi = 0 \tag{1}$$

has the feature that if ψ is as solution, so is $\gamma_5\psi$:

$$\not{D}(\gamma_5) = 0. \tag{2}$$

The matrices

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5) \tag{3}$$

are projectors,

$$P_\pm^2 = P_\pm \quad P_+P_- = P_-P_+ = 0. \tag{4}$$

To understand the physical significance of these projectors, it is convenient to use a particular basis for the Dirac matrices, often called the Chiral or Weyl basis:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{5}$$

where

$$\sigma^\mu = (1, \vec{\sigma}) \qquad \bar{\sigma}^\mu = (1, -\vec{\sigma}). \tag{6}$$

In this basis,

$$\gamma_5 = i\gamma^o\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7}$$

so

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8}$$

We will adopt some notation, following the text by Wess and Bagger:

$$\psi = \begin{pmatrix} \chi_\alpha \\ \phi^{*\dot{\alpha}} \end{pmatrix}. \tag{9}$$

Correspondingly, we label the indices on the matrices σ^μ and $\bar{\sigma}^\mu$ as

$$\sigma^\mu = \sigma^\mu_{\alpha\dot{\alpha}} \qquad \bar{\sigma}^\mu = \bar{\sigma}^{\mu\beta\dot{\beta}}. \tag{10}$$

This allows us to match upstairs and downstairs indices, and will prove quite useful. The Dirac equation now becomes:

$$i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\phi^{*\dot{\alpha}} = 0 \qquad i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\chi_\alpha = 0. \tag{11}$$

χ and ϕ^* are equivalent representations of the Lorentz group. χ and ϕ obey identical equations. Complex conjugating the second equation in eqn. 11, and noting $\sigma_2\sigma^{\mu*}\sigma_2 = \bar{\sigma}^\mu$.

Before discussing this identification in terms of representations of the Lorentz group, it is helpful to introduce some further notation. First, we define complex conjugation to change dotted to undotted indices. So, for example,

$$\phi^{*\dot\alpha} = (\phi^\alpha)^*. \tag{12}$$

Then we define the anti-symmetric matrices $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ by:

$$\epsilon^{12} = 1 = -\epsilon^{21} \qquad \epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}. \tag{13}$$

The matrices with dotted indices are defined identically. Note that, with upstairs indices, $\epsilon = i\sigma_2$, $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta^\gamma_\alpha$. We can use these matrices to raise and lower indices on spinors. Define $\phi_\alpha = \epsilon_{\alpha\beta}\phi^\beta$, and similar for dotted indices. So

$$\phi_\alpha = \epsilon_{\alpha\beta}(\phi^{*\dot\beta}) * . \tag{14}$$

Finally, we will define complex conjugation of a product of spinors to invert the order of factors, so, for example, $(\chi_\alpha\phi_\beta)^* = \phi^*_{\dot\beta}\chi^*_{\dot\alpha}$.

With this in hand, the reader should check that the action for our original four component spinor is:

$$\begin{aligned} S &= \int d^4x\mathcal{L} = \int d^4x \left(i\chi_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu\chi_\alpha + i\phi^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\phi^{*\dot\alpha}\right) \\ &= \int d^4x\mathcal{L} = \int d^4x \left(i\chi^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\chi^{*\dot\alpha} + i\phi^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\phi^{*\dot\alpha}\right). \end{aligned} \tag{15}$$

At the level of Lorentz-invariant lagrangians or equations of motion, there is *only one* irreducible representation of the Lorentz algebra for massless fermions.

Two component fermions have definite helicity. For a single-particle state with momentum $\vec p = p\hat z$, the Dirac equation reads:

$$p(1-\sigma_z)\phi = 0. \tag{16}$$

Similarly, the reader should check that the anti-particle has the opposite helicity.

It is instructive to describe quantum electrodynamics with a massive electron in two-component language. Write

$$\psi = \begin{pmatrix} e \\ \bar e^* \end{pmatrix}. \tag{17}$$

In the lagrangian, we need to replace ∂_μ with the covariant derivative, D_μ. e contains annihilation operators for the left-handed electron, and creation operators for the corresponding anti-particle. $\bar e$ contains annihilation operators for a particle with the opposite helicity and charge of e, and $\bar e^*$, and creation operators for the corresponding antiparticle.

The mass term, $m\bar\psi\psi$, becomes:

$$m\bar\psi\psi = me^\alpha\bar e_\alpha + me^*_{\dot\alpha}\bar e^{*\dot\alpha}. \tag{18}$$

Again, note that both terms preserve electric charge. Note also that the equations of motion now couple e and $\bar e$.

It is helpful to introduce one last piece of notation. Call

$$\psi\chi = \psi^\alpha \chi_\alpha = -\psi_\alpha \chi^\alpha = \chi^\alpha \psi_\alpha = \chi\psi. \tag{19}$$

Similarly,

$$\psi^* \chi^* = \psi^*_{\dot\alpha} \chi^{*\dot\alpha} = -\psi^{*\dot\alpha} \chi^*_{\dot\alpha} \chi^*_{\dot\alpha} \psi^{*\dot\alpha} = \chi^* \psi^*. \tag{20}$$

Finally, note that with these definitions,

$$(\chi\psi)^* = \chi^* \psi^*. \tag{21}$$