Supersymmetry: Fundamentals

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Supersymmetry (SUSY) is a symmetry which relates fermions and bosons, i.e. fields with different spin. That such a thing exists is itself a surprise (Coleman-Mandula). One suspects an intimate connection with the structure of space-time.

For physics, susy is interesting because of

1. The *hierarchy problem*: why is the Higgs mass much smaller than other scales of physics? This points to supersymmetry at TeV energy scales.

2. TeV supersymmetry yields unification of gauge couplings.

3. TeV supersymmetry can account for the observed dark matter density.

4. More theoretically, supersymmetry emerges frequently from string theory, perhaps somehow important in a fundamental theory of gravity (but this argument doesn't suggest a scale).

The Hierarchy Problem

Dimensional analysis: $m_h \sim M_p$.

Much smaller: symmetries?

But in ordinary field theory, no symmetry can prevent $m_h^2|h|^2$. Radiative corrections in the Standard Model would seem to generate corrections of just this order, even if somehow absent classically.

Other failures of dimensional analysis are familiar in particle physics: m_p, m_e. For m_e, explanation is chiral symmetry: if exact, forbids a mass; breaking proportional to m_e. m_p finds its explanation in QCD dynamics: $m_p \propto M_p e^{-8\pi^2/b_0 g^2(M_p)}$.

Supersymmetry: because bosons are related to fermions, it is possible to account for failure of dimensional analysis due to symmetries and approximate symmetries.

Goal and Plan for these two lectures:

1. Basics of supersymmetry in four dimensions: construction of lagrangians with global supersymmetry.

2. Some features of the quantum theory: non-renormalization theorems

3. Breaking of supersymmetry: spontaneous breaking, Goldstino's theorem

4. Soft breaking of supersymmetry

5. Description of lagrangians with local supersymmetry.

Notation: Two-component spinors

(More detailed notes on web)

We will adopt some notation, following the text by Wess and Bagger:

$$\psi = \begin{pmatrix} \chi_\alpha \phi^{*\dot{\alpha}} \end{pmatrix}.$$

Correspondingly, we label the indices on the matrices σ^μ and $\bar{\sigma}^\mu$ as

$$\sigma^\mu = \sigma^\mu_{\alpha\dot{\alpha}} \qquad \bar{\sigma}^\mu = \bar{\sigma}^{\mu\beta\dot{\beta}}.$$

This allows us to match upstairs and downstairs indices, and will prove quite useful. The Dirac equation now becomes:

$$i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\phi^{*\dot{\alpha}} = 0 \qquad i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\chi_\alpha = 0.$$

χ and ϕ^* are equivalent representations of the Lorentz group. χ and ϕ obey identical equations. Complex conjugating the second equation in eqn. , and noting $\sigma_2\sigma^{\mu*}\sigma_2 = \bar{\sigma}^\mu$.

Define complex conjugation to change dotted to undotted indices. So, for example,

$$\phi^{*\dot{\alpha}} = (\phi^{\alpha})^{*}.$$

Define the anti-symmetric matrices $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ by:

$$\epsilon^{12} = 1 = -\epsilon^{21} \qquad \epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}.$$

The matrices with dotted indices are defined identically. Note that, with upstairs indices, $\epsilon = i\sigma_2$, $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_{\alpha}^{\gamma}$. We can use these matrices to raise and lower indices on spinors. Define $\phi_{\alpha} = \epsilon_{\alpha\beta}\phi^{\beta}$, and similar for dotted indices. So

$$\phi_{\alpha} = \epsilon_{\alpha\beta}(\phi^{*\dot{\beta}}) * .$$

Finally, we will define complex conjugation of a product of spinors to invert the order of factors, so, for example, $(\chi_{\alpha}\phi_{\beta})^{*} = \phi^{*}_{\dot{\beta}}\chi^{*}_{\dot{\alpha}}$.

It is helpful to introduce one last piece of notation. Call

$$\psi\chi = \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi.$$

Similarly,

$$\psi^*\chi^* = \psi^*_{\dot\alpha}\chi^{*\dot\alpha} = -\psi^{*\dot\alpha}\chi^*_{\dot\alpha}\chi^*_{\dot\alpha}\psi^{*\dot\alpha} = \chi^*\psi^*.$$

Finally, note that with these definitions,

$$(\chi\psi)^* = \chi^*\psi^*.$$

The Supersymmetry Algebra and its Representations

Because the supersymmetry generators are spinors, they do not commute with the Lorentz generators. The supersymmetry algebra involves the translation generators. For $N = 1$ supersymmetry:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0.$$

One can have more supersymmetries than this – as many as eight. But while these theories with *extended supersymmetry* are important theoretically, there are several reasons to think that only $N = 1$ can be relevant to the world around us (chiral fermions, supersymmetry breaking.

Representations of the $N = 1$ algebra:

There are three irreducible representations of $N = 1$ supersymmetry which can describe massless fields:

- Chiral superfields fields: (ϕ, ψ_α), a complex fermion and a chiral scalar

- Vector superfields: (λ, A_μ), a chiral fermion and a vector meson, both, in general, in the adjoint representation of the gauge group

- The gravity supermultiplet: $(\psi_{\mu,\alpha}, g_{\mu\nu})$, a spin-3/2 particle, the gravitino, and the graviton.

One can work in terms of these component fields, writing supersymmetry transformation laws and constructing invariants. This turns out to be rather complicated. One must use the equations of motion to realize the full algebra. Great simplification is achieved by enlarging space-time to include commuting and anti-commuting variables. The resulting space is called *superspace*.

Superspace

Superspace coordinates:

$$x^\mu,\ \theta_\alpha, \theta^*_{\dot\alpha} = \bar\theta_{\dot\alpha}$$

The Grassman coordinates obey:

$$\{\theta_\alpha, \theta_\beta\} = \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = \{\theta_\alpha, \bar\theta_{\dot\beta}\} = 0.$$

Note $\theta_1^2 = \theta_2^2 = \ldots = 0$. Grassman coordinates are familiar from the problem of formulating the fermion functional integral; they provide a representation of the classical configuration space for fermions.

Because the square of any θ vanishes, functions of Grassman variables are polynomials. Derivatives anticommute:

$$\{\frac{\partial}{\partial\theta_\alpha}, \frac{\partial}{\partial\bar\theta_{\dot\beta}}\} = 0.$$

Integration of Grassman variables

For Poincare invariance of ordinary field theory lagrangians, important that:

$$\int_{-\infty}^{\infty} dx f(x+a) = \int_{-\infty}^{\infty} dx f(x)$$

For Grassman integration (one variable):

$$\int d\theta f(\theta+\epsilon) = \int d\theta f(\theta)$$

Satisfied by the integral table:

$$\int d\theta (1,\theta) = (0,1)$$

For the case of θ_α,$\bar{\theta}_{\dot{\alpha}}$, one can write a simple integral table:

$$\int d^2\theta \theta^2 = 1; \int d^2\bar{\theta}\bar{\theta}^2 = 1,$$

all others vanishing.

In superspace, can provide a classical description of the action of the symmetry on fields. On functions $f(x^\mu, \theta, \bar{\theta})$ the supersymmetry generators act as differential operators: x^μ, θ, θ^*:

$$Q_\alpha = \frac{\partial}{\partial\theta_\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu; \qquad \bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu.$$

The θ's have mass dimension $-1/2$.

These differential operators obey the susy algebra. E.g.

$$\{Q_\alpha, Q_\beta\} = \{(\frac{\partial}{\partial\theta_\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu), (\frac{\partial}{\partial\theta_\beta} - i\sigma^\nu_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\nu)\} = 0$$

since the θ's and their derivatives anticommute.

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu.$$

The Q's generate infinitesimal transformations in superspace with parameter ϵ. One can construct finite transformations as well by exponentiating the Q's; because there are only a finite number of non-vanishing polynomials in the θ's, these exponentials contain only a finite number of terms. The result can be expressed compactly:

$$e^{\epsilon Q+\epsilon^* Q^*}\Phi(x^\mu,\theta,\bar\theta) = \Phi(x^\mu - i\epsilon\sigma^\mu\theta^* + i\theta\sigma^\mu\epsilon^*, \theta+\epsilon, \bar\theta+\epsilon^*).$$

Irreducible Representations

A superfield Φ can be decomposed into two irreducible representations of the algebra, corresponding to the chiral and vector superfields described above. We need one more set of objects, the covariant derivatives, D_α and $\bar{D}_{\dot{\alpha}}$.

$$D_\alpha = \frac{\partial}{\partial\theta_\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu; \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu.$$

They satisfy the anticommutation relations:

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \qquad \{D_\alpha, D_\alpha\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0.$$

The D's anticommute with the Q's:

$$\{D_\alpha, \bar{Q}_{\dot{\alpha}}\} = 0$$

etc.

Chiral Superfields

Satisfy:

$$\bar{D}_{\dot{\alpha}}\Phi = 0$$

Because the D's anticommute with the Q's, this condition is invariant under supersymmetry transformations.

Construction:

$$y = x^\mu + i\theta\sigma^\mu\overline{\theta}.$$

so $\bar{D}y = 0$, and

$$\Phi = \Phi(y) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y).$$

is a chiral (scalar) superfield. Expanding in θ:

$$\Phi = \phi(x) + i\theta\sigma^\mu\overline{\theta}\partial_\mu\phi + \frac{1}{4}\theta^2\overline{\theta}^2\partial^2\phi$$

$$+ \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\overline{\theta} + \theta^2 F.$$

Transformation Laws for the Components of a Chiral Superfield:

$$\delta\Phi = \epsilon_\alpha Q^\alpha \Phi$$

so matching the coefficients of powers of θ on each side, the components transform as

$$\delta\phi = \sqrt{2} i\epsilon\psi \qquad \delta\psi = \sqrt{2}\epsilon F + \sqrt{2} i\sigma^\mu \epsilon^* \partial_\mu \phi \qquad \delta F = 0$$

Vector superfields

Satisfy the condition

$$V = V^\dagger$$

Preserved by SUSY transformations:

$$\delta V = (\epsilon Q + \bar{\epsilon}\bar{Q})V \qquad \delta V = \delta V^\dagger$$

V can be expanded in a power series in θ's:

$$V = i\chi - i\chi^\dagger - \theta\sigma^\mu\theta^* A_\mu + i\theta^2\overline{\theta}\overline{\lambda} - i\overline{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\overline{\theta}^2 D.$$

χ is a chiral field.

If V is to describe a massless field, the presence of A_μ indicates that there should be some underlying gauge symmetry, which generalizes the conventional transformation of bosonic theories. In the case of a $U(1)$ theory, gauge transformations act by

$$V \to V + i\Lambda - i\Lambda^\dagger$$

where Λ is a chiral field. The $\theta\bar{\theta}$ term in Λ ($\Lambda \sim \theta\sigma^\mu\bar{\theta}\partial\mu\chi$) is precisely a conventional gauge transformation of A_μ. One can define a gauge-invariant field strength,

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V.$$

$$\Delta W_\alpha = -\frac{i}{4}\bar{D}^2 D_\alpha(\Lambda - \Lambda^\dagger)$$

$$= -\frac{i}{4}\bar{D}^2 D_\alpha \Lambda$$

$$= -\frac{i}{4}\bar{D}^{\dot{\alpha}}\{\bar{D}_{\dot{\alpha}}, D_\alpha\}\Lambda = 0.$$

The field content of the vector field: Wess-Zumino gauge

By a gauge transformation, we can set $\chi = 0$. The resulting gauge is known as the Wess-Zumino gauge. This gauge is analogous to Coulomb gauge in electrodynamics: the degrees of freedom are clear, but the gauge condition breaks supersymmetry.

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D - \sigma^{\mu\nu}{}_\alpha^{\beta} F_{\mu\nu}\theta_\beta + \theta^2 \sigma^\mu_{\alpha\dot{\beta}} \partial_\mu \lambda^{*\dot{\beta}}.$$

The gauge transformation of a chiral field of charge q is:

$$\Phi \to e^{-iq\Lambda}\Phi$$

One can form gauge invariant combinations using the vector field (connection) V:

$$\Phi^\dagger e^{qV}\Phi.$$

We can also define a *gauge-covariant derivative* by

$$\mathcal{D}_\alpha \Phi = D_\alpha \Phi + D_\alpha V \Phi.$$

Non-Abelian Gauge Theories

Generalize first the transformation of Φ:

$$\Phi \to e^{-i\Lambda}\Phi$$

where Λ is a matrix-valued chiral field.

Introduce a matrix-valued field, V, and require that

$$\phi^\dagger e^V \phi$$

be gauge invariant. So we require:

$$e^V \to e^{-i\Lambda^*} e^V e^{i\Lambda^*}.$$

From this, we can define a gauge-covariant field strength,

$$W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V.$$

This transforms like a chiral field in the adjoint representation:

$$W_\alpha \to e^{i\Lambda} W_\alpha e^{-i\Lambda}.$$

N=1 Lagrangians

In ordinary field theories, we construct lagrangians invariant under translations by integrating densities over all of space. The lagrangian changes by a derivative under translations, so the *action* is invariant. Similarly, if we start with a lagrangian density in superspace, a supersymmetry transformation acts by differentiation with respect to x or θ. So integrating the variation over the full superspace gives zero. This is the basic feature of the integration rules we introduced earlier. In equations:

$$\delta \int d^4x \int d^4\theta \; h(\Phi, \Phi^\dagger, V)$$

$$= \int d^4x d^4\theta \; (\epsilon_\alpha Q^\alpha + \epsilon_{\dot\alpha} Q^{\dot\alpha}) h(\Phi, \Phi^\dagger, V) = 0.$$

For chiral fields, integrals over *half* of superspace are invariant. If $f(\Phi)$ is a function of chiral fields only, f itself is chiral. As a result,

$$\delta \int d^4x \; d^2\theta f(\Phi) = \int d^4x \; d^2\theta (\epsilon_\alpha Q^\alpha + \epsilon_{\dot\alpha} Q^{\dot\alpha}) f(\Phi).$$

The Q_α terms vanish when integrated over x and $d^2\theta$. The Q^* terms also give zero:

$$Q^*_{\dot\alpha} f \propto \theta^\alpha \sigma^\mu{}_{\alpha\dot\alpha} \partial_\mu f.$$

With these ingredients, we can write down the most general renormalizable lagrangian. The chiral fields, Φ_i, have dimension one; the θ's have dimension $-1/2$. The vector fields, $V^{(i)f}$, are dimensionless, while W_α has dimension 3/2. First, there are terms involving integration over the full superspace:

$$\mathcal{L}_{kin} = \int d^4\theta \sum_i \Phi_i^\dagger e^V \Phi_i,$$

where the e^V is in the representation of the gauge group appropriate to the field Φ_i. We can also write an integral over half of superspace:

$$\mathcal{L}_W = \int d^2\theta W(\Phi_i) + \text{c.c.}$$

$W(\Phi)$ is a holomorphic function of the Φ_i's (it is a function of Φ_i, not $\Phi_i^\dagger$), called the superpotential. For a renormalizable theory,

$$W = \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} \Gamma_{ijk} \Phi_i \Phi_j \Phi_k$$

Finally, for the gauge fields, we can write:

$$\mathcal{L}_{gauge} = \frac{1}{g^{(i)2}} \int d^2\theta W_\alpha^{(i)2}$$

The full lagrangian density is

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_W + \mathcal{L}_{gauge}.$$

Component Form of the Lagrangian

To obtain the lagrangian for the component fields, we just use the expressions for the fields and the integration rules for the θ's. First consider a single chiral field, Φ, neutral under any gauge symmetries. Then

$$\mathcal{L}_{kin} = |\partial_\mu \Phi|^2 + i\psi_\Phi \partial_\mu \sigma^\mu \psi_\Phi^* + F_\Phi^* F_\Phi.$$

The field F is referred to as an "auxiliary field," as it appears without derivatives in the action; it has no dynamics. For several fields,

$$\mathcal{L}_{kin} = |\partial_\mu \phi_i|^2 + i\psi_i \partial_\mu \sigma^\mu \psi_i^* + F_i^* F_i.$$

From W (several fields)

$$\mathcal{L}_W = \frac{\partial W}{\partial \Phi_i} F_i + \frac{\partial^2 W}{\partial \Phi_i \Phi_j} \psi_i \psi_j.$$

For our special choice of superpotential this is:

$$\mathcal{L}_W = F_i(m_{ij}\Phi_j + \Gamma_{ijk}\Phi_j\Phi_k) + (m_{ij} + \Gamma_{ijk}\Phi_k)\psi_i\psi_j + \text{c.c.}.$$

It is a simple matter to solve for the auxiliary fields:

$$F_i^* = -\frac{\partial W}{\partial \Phi_i}$$

Substituting back in the lagrangian,

$$V = |F_i|^2 = |\frac{\partial W}{\partial \Phi_i}|^2$$

Lagrangian with the Gauge Fields

To work out the couplings of the gauge fields, it is convenient to choose the Wess-Zumino gauge. Leaving the details for the exercises:

$$\mathcal{L} = -\frac{1}{4} g_a^{-2} F_{\mu\nu}^{a2} - i\lambda^a \sigma^\mu D_\mu \lambda^{a*} + + |D_\mu \phi_i|^2 - i\psi_i \sigma^\mu D_\mu \psi_i^* + \frac{1}{2g^2}(D^a)^2 + D^a$$

$$+ F_i^* F_i - F_i \frac{\partial W}{\partial \phi_i} + \text{cc} + \sum_{ij} \frac{1}{2} \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \psi_i \psi_j + i\sqrt{2} \sum \lambda^a \psi_i T^a \phi_i^*.$$

The scalar potential is found by solving for the auxiliary D and F fields:

$$V = |F_i|^2 + \frac{1}{2g_a^2}(D^a)^2$$

with

$$F_i = \frac{\partial W}{\partial \phi_i^*} \qquad D^a = \sum_i (g^a \phi_i^* T^a \phi_i).$$

The Fayet-Iliopoulos Term

In the case there is a $U(1)$ factor in the gauge group, there is one more term one can include in the lagrangian, known as the Fayet-Iliopoulos D term. In superspace,

$$\int d^4\theta V$$

is supersymmetric and gauge invariant:

$$\delta \int d^4\theta V = i \int d^4\theta(\Lambda - \Lambda^\dagger) = 0.$$

(e.g.

$$\int d\bar{\theta} f = \frac{d}{d\bar{\theta}} f; \quad \frac{d}{d\bar{\theta}} = \bar{D} - \text{(total derivative)}.$$

In components, this is simply a term linear in D, ξD, so, solving for D from its equations of motion,

$$D = \xi + \sum_i q_i \phi_i^* \phi_i.$$

The Supersymmetry Currents

Noether procedure: need to be careful because the lagrangian is not invariant under supersymmetry transformations, but instead transforms by a total derivative (compare translations). Variation of the lagrangian is proportional to $\int d^4\theta \epsilon Q \mathcal{L}$. The piece involving $\frac{\partial}{\partial\theta}$ integrates to zero, but the other piece; only in the action, obtained by integrating the lagrangian density over space-time, does the derivative term drop out.

So in performing the Noether procedure, the variation of the lagrangian will have the form:

$$\delta\mathcal{L} = \epsilon\partial_\mu K^\mu + (\partial_\mu \epsilon) T^\mu$$

Integrating by parts, we have that $K^\mu - T^\mu$ is conserved. Taking this into account, for a theory with a single chiral field:

$$j^\mu_\alpha = i\psi^* \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \phi + F \sigma^\mu_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}},$$

and similarly for $j^\mu_{\dot{\alpha}}$.

For several chiral fields, replace $\psi \to \psi_i$, $\phi \to \phi_i$, etc., and sum over i. One can check that the (anti)commutator of the Q's (integrals over j^o) with the various fields gives the correct transformations laws. E.g.:

$$\delta\psi(x) = \epsilon^\alpha \{Q_\alpha, \psi(x)\}$$

$$= i\epsilon^\alpha \sigma^\mu_{\alpha,\cdot\alpha} \partial_\mu \phi + \epsilon^\alpha F.$$

One can do the same for gauge fields. For gauge fields, working with the action written in terms of W, there are no derivatives, so the variation of the lagrangian comes entirely from the $\partial_\mu K^\mu$ term. Working out the current is left for the exercises.

The Ground State Energy as an Order Parameter for Supersymmetry Breaking

In globally supersymmetric lagrangians, $V \geq 0$. This fact can be traced back to the supersymmetry algebra.

$$\{Q_\alpha, Q_{\dot\beta}\} = 2P_\mu \sigma^\mu_{\alpha\dot\beta},$$

multiply by σ^o and take the trace:

$$Q_\alpha Q_{\dot\alpha} + Q_{\dot\alpha} Q_\alpha = E.$$

The left hand side is positive definite, so the energy is always greater than or equal to zero.

$E = 0$ is special: the expectation value of the energy is an *order parameter* for supersymmetry breaking. If supersymmetry is unbroken, $Q_\alpha|0\rangle = 0$, so the ground state energy vanishes *if and only if* supersymmetry is unbroken.

The Auxiliary Fields F and D as Order Parameters for Supersymmetry Breaking

Alternatively, consider the supersymmetry transformation laws for λ and ψ. One has, under a supersymmetry transformation with parameter ϵ,

$$\delta\psi = \sqrt{2}\epsilon F + \dots \qquad \delta\lambda = i\epsilon D + \dots$$

In the quantum theory, the supersymmetry transformation laws become operator equations:

$$\delta\psi = \{Q, \psi\}$$

so taking the vacuum expectation value of both sides, we see that a non-vanishing F means broken supersymmetry; again vanishing of the energy, or not, is an indicator of supersymmetry breaking. So if either F or D has an expectation value, supersymmetry is broken.

The signal of ordinary (bosonic) symmetry breakdown is a Goldstone boson. In the case of supersymmetry, the signal is the presence of a Goldstone fermion, or goldstino. One can prove a goldstino theorem in almost the same way one proves Goldstone's theorem. We will do this shortly, when we consider simple models of supersymmetry and its breaking.

The Wess-Zumino Model

One of the earliest, and simplest, models is the Wess-Zumino model, a theory of a single chiral field (no gauge interactions). For the superpotential, we take:

$$W = \frac{1}{2}m\phi^2 + \frac{\lambda}{3}\phi^3$$

The scalar potential is:

$$V = |m\phi + \lambda\phi^2|^2$$

and the ϕ field has mass-squared $|m|^2$. The fermion mass term is

$$\frac{1}{2}m\psi\psi$$

so the fermion also has mass m.

Symmetries of the Wess-Zumino Model

First, set $m = 0$. The theory then has a continuous global symmetry. This is perhaps not obvious from the form of the superpotential, $W = \frac{\lambda}{3}\phi^3$. But the lagrangian is an integral over superspace of W, so it is possible for ϕ to transform and for the θ's to transform in a compensating fashion. Such a symmetry, which does not commute with supersymmetry, is called an R symmetry. If, by convention, we take the θ's to carry charge 1, than the $d\theta$'s carry charge -1 (think of the integration rules). So the superpotential must carry charge 2. In the present case, this means that ϕ carries charge 2/3. Note that each component of the superfield transforms differently:

$$\phi \rightarrow e^{i\frac{2}{3}\alpha}\phi \qquad \psi \rightarrow e^{i(\frac{2}{3}-1)\alpha}\psi \qquad F \rightarrow e^{i(\frac{2}{3}-2)\alpha}F$$

A U(1) Gauge Theory

Consider a $U(1)$ gauge theory, with two charged chiral fields, ϕ^+ and ϕ^-, with charges ± 1, respectively. Suppose $W = 0$. In Wess-Zumino gauge:

$$V(\phi^\pm) = \frac{1}{2}D^2 = \frac{g^2}{2}(|\phi^+|^2 - |\phi^-|^2)^2.$$

Zero energy, supersymmetric minima have $D = 0$. By a gauge choice, we can set

$$\phi = v \qquad \phi^- = v' e^{i\alpha}$$

Then $D = 0$ if $v = v'$. Each vacuum is physically distinct – in this example the spectra are different – and there are no transitions between vacua.

Spectrum:

- Gauge bosons, with masses:

  $$m_v^2 = 4g^2v^2.$$

  This accounts for three degrees of freedom.

- Gauginos: From the Yukawa couplings of the gaugino, λ, to the ϕ's:

  $$\mathcal{L}_\lambda = \sqrt{2}g\lambda(\psi_{\phi^+} - \psi_{\phi_-})$$

  so we have a Dirac fermion with mass $2gv$. So we now have accounted for three bosonic and four fermionic degrees of freedom, all degenerate.

- The fourth bosonic degree of freedom is a scalar,the "partner" of the Higgs which is eaten in the Higgs phenomenon. To compute its mass, note that, expanding the scalars as

$$\phi^{\pm} = v + \delta\phi^{\pm}$$

$$D = 2v(\delta\phi^{+} + \delta\phi^{+*} - \delta\phi^{-} - \delta\phi^{-*})$$

So $\frac{1}{2}D^2$ gives a mass to the real part of $1/\sqrt{2}(\delta\phi^{+} - \delta\phi^{-})$, equal to the mass of the gauge bosons and gauginos. Since the masses differ in states with different v, these states are physically inequivalent.

- There is also a massless state: a single chiral field. For the scalars, this follows on physical grounds. First, there should be a Goldstone boson due to the breaking of the symmetry $\phi^{\pm} \rightarrow e^{i\alpha}\phi^{\pm}$. Another scalar arises because the expectation value, v, is undetermined. For the fermions, the linear combination $\psi_{\phi^+} + \psi_{\phi^-}$ is not fixed. So we have the correct number of fields to construct a massless chiral multiplet. We can describe this elegantly by introducing the composite chiral superfield:

$$\Phi = \phi^{+}\phi^{-} \approx v^2 + v(\delta\phi^{+} + \delta\phi^{-}).$$

Its components are precisely the massless complex scalar and chiral fermion which we identified above.

Moduli

This is our first encounter with a phenomena which is nearly ubiquitous in supersymmetric field theories: there are often continuous sets of vacuum states, at least in some approximation. The set of such physically distinct vacua is known as the "moduli space." In this example, the set of such states is parameterized by the values of the field, Φ; Φ is called a "modulus."

In quantum mechanics, in such a situation, we would solve for the wave function of the modulus, and the ground state would typically involve a superposition of the different classical ground states. We have seen, though, that, in field theory, one must choose a value of the modulus field. In the presence of such a degeneracy, for each such value one has, in effect, a different theory – no physical process leads to transitions between one such state and another. Once the degeneracy is lifted, however, this is no longer the case, and transitions, as we will frequently see, are possible.

General Supersymmetric Lagrangian and Non-Renormalization Theorems

So far, renormalizable field theories. The most general, globally supersymmetric theory with at most two derivatives:

$$\mathcal{L} = \int d^4\theta K(\phi_i, \phi_i^\dagger) + \int d^2\theta W(\phi_i) + \text{c.c.}$$

$$+ \int d^2\theta f_a(\phi)(W_\alpha^{(a)})^2 + \text{c.c.}.$$

K: Kahler potential. W and f_a are holomorphic.

Re $f\ F_{\mu\nu}^2 +$ Imf $F\ \tilde{F}$.

Non-supersymmetric theories have the property that they tend to be generic; any term permitted by symmetries in the theory will appear in the effective action, with an order of magnitude determined by dimensional analysis. Not so for susy theories.The superpotential is not corrected in perturbation theory beyond its tree level value; f is at most renormalized at one loop.

Seiberg argued that the coupling constants of a theory may be thought of as *expectation values* of chiral fields and so the superpotential must be a holomorphic function of these as well.

E.g. consider a theory of a single chiral field, Φ, with superpotential

$$W = \int d^2\theta (m\Phi^2 + \lambda\Phi^3).$$

We can think of λ and m as expectation values of chiral fields, $\lambda(x,\theta)$ and $m(x,\theta)$.

In the Wess-Zumino lagrangian there is an R symmetry under which Φ has R charge one and λ has R charge -1 (m has R charge 0). Now consider corrections to the effective action in perturbation theory. Renormalizations of λ in the superpotential necessarily involve positive powers of λ. But such terms (apart from $(\lambda)^1$) have the wrong R charge to preserve the symmetry. So there can be no renormalization of this coupling. There can be wave function renormalization, since K is not holomorphic, so $K = K(\lambda^\dagger\lambda)$ is allowed, in general.

This non-renormalization of the superpotential is general (first proven by detailed studies of Feynman diagrams).

Other Non-Renormalization Theorems

Can treat g^{-2} as part of a chiral field.

$$S = \frac{8\pi^2}{g^2} + ia + \ldots \qquad \int d^2\theta S W_\alpha^2$$

The real part of the scalar field in this multiplet couples to $F_{\mu\nu}^2$, but the imaginary part, a, couples to $F\tilde{F}$. $F\tilde{F}$ is a total derivative, so in perturbation theory there is a symmetry under constant shifts of a. The effective action should respect this symmetry. Because the gauge coupling function, f, is holomorphic, this implies that

$$f(g^2) = S + \text{const} = \frac{8\pi^2}{g^2} + \text{const} \ .$$

The first term is just the tree level term. One loop corrections yield a constant, but there are no higher order corrections in perturbation theory! This is quite a striking result. It is also paradoxical, since the two loop β-functions for supersymmetric Yang-Mills theories have been computed long ago, and are in general non-zero. The resolution of this paradox is subtle and interesting.

Fayet-Iliopoulos term

If there is no Fayet-Iliopoulos D-term at tree level, this term can be generated at most at one loop. To prove this, write the D term as

$$\int d^4\theta d(g,\lambda)V.$$

$d(g,\lambda)$ is some unknown function of the gauge and other couplings in the theory. But if we think of g and λ as chiral fields, this expression is only gauge invariant if d is a constant, corresponding to a possible one loop contribution.

Spontaneous Supersymmetry Breaking: O'Raifeartaig Models

Supersymmetry breaking is signalled by a non-zero expectation value of an F component of a chiral or D component of a vector superfield. Models involving only chiral fields with no supersymmetric ground state are called O'Raifeartaigh models. E.g. fields, A, B, and X, with superpotential:

$$W = \lambda A(X^2 - \mu^2) + mBX.$$

The equations

$$F_A = \lambda(x^2 - \mu^2) = 0 \qquad F_B = mX = 0$$

are incompatible. There is no problem satisfying the equation $F_X = 0$. So we need to minimize

$$V_{eff} = |F_A|^2 + |F_B|^2 = |\lambda^2||X^2 - \mu^2|^2 + m^2|X|^2.$$

Assuming μ^2, λ are real, the solutions are:

$$X = 0 \qquad X^2 = \frac{2\lambda^2\mu^2 - m^2}{2\lambda^2}$$

The corresponding vacuum energies are:

$$V_o = \lambda^2\mu^4 \qquad \frac{m^2\mu^2}{\lambda^2} - \frac{m^4}{4\lambda^2}.$$

The vacuum at $X \neq 0$ disappears at a critical value of μ.

Spectrum in the $X = 0$ vacuum: several massless states First, a massless scalar; arises because, at this level not all of the fields are fully determined. The equation

$$\frac{\partial W}{\partial X} = 0$$

can be satisfied provided

$$2\lambda_1 AX + mB = 0.$$

This vacuum degeneracy is accidental and will be lifted by quantum corrections (Seiberg's lectures).

There is also a massless fermion, ψ_A. This fermion is the Goldstino. Replacing the auxiliary fields in the supersymmetry current for this model, gives

$$j = F_A \sigma^\mu \psi_A^*.$$

Sum Rule

The massive states do not form Bose-Fermi degenerate multiplets. They satisfy a sum rule,

$$\sum (-1)^F m^2 = 0.$$

Here $(-1)^F = 1$ for bosons and -1 for fermions.

$$F_A = -\lambda\mu^2 \quad \text{Take } X = 0.$$

Dirac fermion: $m\psi_X\psi_B$.

Scalar masses from $m^2|B|^2 + m^2|X|^2 - \lambda^2\mu^2(X^2 + X^{*2})$.

Re X: $m^2 - 2\lambda^2\mu^2 = m^2 + 2\lambda F_A$ Im X: $m^2 + 2\lambda^2\mu^2 = m^2 - 2\lambda F_A$

The sum rule is general. Here, proof without gauge interactions. The potential is given by

$$V = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 .$$

The boson mass matrix has terms of the form $\phi_i^* \phi_j$ and $\phi_i \phi_j + \text{c.c.}$ (indices $\bar{i}$ and $\bar{j}$ for complex conjugate fields). The latter terms are connected with supersymmetry breaking. The various terms in the mass matrix can be obtained by differentiating the potential:

$$m^2_{i\bar{j}} = \frac{\partial^2 V}{\partial \phi_i \partial \phi^*_{\bar{j}}} = \frac{\partial^2 W}{\partial \phi_i \partial \phi_k} \frac{\partial^2 W^*}{\partial \phi^*_{\bar{k}} \partial \phi^*_{\bar{j}}},$$

$$m^2_{ij} = \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} = \frac{\partial W}{\partial \phi^*_k} \frac{\partial^3 W}{\partial \phi_k \partial \phi_i \partial \phi_j}.$$

The first of theses terms has just the structure of the square of the fermion mass matrix,

$$\mathcal{M}_{\mathcal{F}ij} = \frac{\partial^2 W}{\partial \phi_i \partial \phi_j}.$$

So writing the boson mass, $\mathcal{M}^2_B$ matrix on the basis $(\phi_i \; \phi^*_{\bar{j}})$, we see that the sum rule holds.

The theorem is true whenever a theory can be described by a renormalizable effective action. We have seen that various non-renormalizable terms in the effective action can give additional contributions to the mass and these will violate the tree level sum rule. Such terms arise in renormalizable theories when one integrates out heavy fields to obtain an effective action at some scale. In the context of supergravity, such terms are present already at tree level.

SUSY can also be broken by expectation value for a D term. E.g. if $\mathcal{L}$ includes

$$\int d^4\theta \xi^2 V + \int d^2\theta m \phi_+ \phi_-$$

$$D(|\phi_+|^2 - |\phi_-|^2 - \xi^2)$$

$$V = m^2(|\phi_+|^2 + |\phi_-|^2) + \frac{1}{2}(|\phi_+|^2 - |\phi_-|^2 - \xi^2)^2$$

and $D \neq 0$ at the minimum. The sum rule holds for these as well.

The Goldstino Theorem

Similar to proof of ordinary Goldstone theorem.. Suppose that the symmetry is broken by the F component of a chiral field (this can be a composite field). Then we can study

$$\int d^4x \partial_\mu e^{iq\cdot x} T < j^\mu_\alpha(x)\psi_\Phi(0) >= 0.$$

j^μ_α is the supersymmetry current; its integral over space is the supersymmetry charge. This expression vanishes because it is an integral of a total derivative. Now taking the derivatives, there are two non-vanishing contributions: one from the derivative acting on the exponential; one from the action on the time-ordering symbol. Taking these derivatives, and then taking the limit $q \to 0$, gives

$$< \{Q, \Psi_\Phi(0)\} >= iq_\mu T < j^\mu_\alpha(x)\psi_\Phi(0) >_{F.T.} .$$

Now the left hand side is constant, so the Green's function on the right hand side must be singular as $q \to 0$. By the usual spectral representation analysis, this shows that there is a massless fermion coupled to the supersymmetry current. Recalling the form of the supersymmetry current, if one of the F's has an expectation value,

$$j^{\mu}_{\alpha} = \sigma_{\mu} \psi^{*}_{\alpha} F.$$

F, here, is the "Goldstino decay constant." We can understand the massless fermion which appeared in the O'Raifeartaigh model in terms of this theorem. It is easy to check that:

$$\psi_G \propto F_A \psi_A + F_B \psi_B.$$

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Explicit, Soft Supersymmetry Breaking

Ultimately, if nature is supersymmetric, it is likely that we will want to understand supersymmetry breaking through some dynamical mechanism. But we can be more pragmatic, accept that supersymmetry is broken, and parameterize the breaking through mass-differences between ordinary fields and their superpartners. It turns out that this procedure does not spoil the good ultraviolet properties of the theory. Such mass terms are said to be "soft" for precisely this reason.

Wess-Zumino model, with $m = 0$. Add term $m^2_{soft}|\phi|^2$. Ask what is the form of corrections to the scalar mass. Without the soft breaking term, there is an exact cancellation of the two Feynman graphs:

$$\delta m^2 = \frac{1}{(2\pi)^4}\int d^4k\left(\frac{1}{k^2} - \frac{1}{k^2}\right)$$

with the soft breaking term, there is not an exact cancellation;

$$\frac{1}{(2\pi)^4}\int d^4k\left(\frac{1}{k^2 + m^2_{soft}} - \frac{1}{k^2}\right)$$

$$= -\frac{|\lambda|^2}{16\pi^2} m^2_{soft} \ln(\Lambda^2/m^2_{soft}).$$

We can understand this simply on dimensional grounds. We know that for $m^2_{soft} = 0$, there is no correction. Treating the soft term as a perturbation, the result is necessarily proportional to m^2_{soft}; at most, then, any divergence must be logarithmic.

Can also add soft masses for gauginos and trilinear scalar couplings

$$m_\lambda \lambda\lambda \qquad A_{ijk}\phi_i\phi_j\phi_k$$

Can understand how these might arise at a more fundamental level (makes clear the sense in which these terms are soft).

$$Z: \quad \langle F_Z \rangle = 0.$$

No renormalizable couplings between Z and ϕ ("matter fields"). Such couplings might be forbidden by symmetries. Non-renormalizable couplings, e.g. in Kahler potential:

$$\mathcal{L}_Z = \frac{1}{M^2}\int d^4\theta Z^\dagger Z \phi^\dagger \phi$$

can be expected to arise in the effective lagrangian; not forbidden by any symmetry. Replacing Z by its expectation value, $< Z >= \ldots + \theta^2 < F_Z >$ gives a mass term for the scalar component of ϕ:

$$\mathcal{L}_Z = \frac{|< F >|^2}{M^2}$$

the soft mass term. Simple power counting shows that loop corrections to these couplings due to renormalizable interactions are at most logarithmically divergent.

The operator:

$$\int d^2\theta \frac{Z}{M} W_\alpha^2 = \frac{F_Z}{M}\lambda\lambda + \dots$$

gives rise to a mass for gauginos. The term

$$\int d^2\theta \frac{Z}{M}\phi\phi\phi$$

leads to a trilinear coupling of the scalars.

To summarize, there are three types of soft breaking terms which can appear in a low energy effective action:

- Soft scalar masses, $m_\phi^2|\phi|^2$.
- Gaugino masses, $m_\lambda\lambda\lambda$
- Trilinear scalar couplings, $\Gamma\phi\phi\phi$.

Local Supersymmetry

If supersymmetry has anything to do with nature, and if it is not merely an accident, then it must be a local symmetry. In addition to chiral and vector fields, now has a graviton and a gravitino.

As in global supersymmetry (without the restriction of renormalizability), the terms in the effective action with at most two derivatives or four fermions are completely specified by three functions:

1. The Kahler potential, $K(\phi, \phi^\dagger)$, a function of the chiral fields

2. The superpotential, $W(\phi)$, a holomorphic function of the chiral fields.

3. The gauge coupling functions, $f^a(\phi)$, which are also holomorphic functions of the chiral fields.

The lagrangian which follows from these is quite complicated, including many two and four fermions interactions. The scalar potential is rather simple and very useful:

$$V = e^K \left[\left(\frac{\partial W}{\partial \phi_i} + \frac{\partial K}{\partial \phi_i} W \right) g^{i\bar{j}} \left(\frac{\partial W^*}{\partial \phi_j^*} + \frac{\partial K}{\partial \phi_j^*} W \right) - 3|W|^2 \right],$$

where

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial \phi_i \partial \phi_{\bar{j}}}$$

is the (Kahler) metric associated with the Kahler potential. In this equation, we have adopted units in which $M_p = 1$, where

$$G_N = \frac{1}{8\pi M_p^2}.$$

$M_p \approx 2 \times 10^{18}$ GeV is known as the *reduced Planck mass*.

Supersymmetry Breaking in Supergravity Models

Supergravity potential:

$$V = e^K \left[\left(\frac{\partial W}{\partial \phi_i} + \frac{\partial K}{\partial \phi_i} W \right) g^{i\bar{j}} \left(\frac{\partial W^*}{\partial \phi_j^*} + \frac{\partial K}{\partial \phi_j^*} W \right) - 3|W|^2 \right],$$

In supergravity, the condition for unbroken supersymmetry is that the *Kahler derivative* of the superpotential should vanish:

$$D_i W = \frac{\partial W}{\partial \phi_i} + \frac{\partial K}{\partial \phi_i} W = 0.$$

When this is not the case, supersymmetry is broken. If we require vanishing of the cosmological constant, then we have:

$$3|W|^2 = \sum_{i,\bar{j}} D_i W D_{\bar{j}} W^* g^{i\bar{j}}.$$

In this case, the gravitino mass turns out to be:

$$m_{3/2} = \langle e^{K/2} W \rangle.$$

There is a standard strategy for building supergravity models. One introduces two sets of fields, the "hidden sector fields," which will be denoted by Z_i, and the "visible sector fields," denoted y_a. The Z_i's are assumed to be connected with supersymmetry breaking, and to have only very small couplings to the ordinary fields, y_a. In other words, one assumes that the superpotential, W, has the form

$$W = W_z(Z) + W_y(y),$$

at least up to terms suppressed by $1/M$. The y fields should be thought of as the ordinary matter fields and their superpartners.

One also usually assumes that the Kahler potential has a "minimal" form,

$$K = \sum z_i^\dagger z_i + \sum y_a^\dagger y_a.$$

One chooses (tunes) the parameters of W_Z so that

$$\langle F_Z \rangle \approx M_w M$$

and

$$\langle V \rangle = 0.$$

Note that this means that

$$\langle W \rangle \approx M_W M_p^2.$$

The simplest model of the hidden sector is known as the "Polony model." In this model,

$$W = m^2(z + \beta)$$

$$\beta = (2 + \sqrt{3}M) \ .$$

The minimum of the potential for Z lies at

$$Z = (\sqrt{3} - 1)M$$

and

$$m_{3/2} = (m^2/m)e^{(\sqrt{3}-1)^2/2}.$$

Soft-breaking mass terms for the fields y: There are terms of the form

$$m_o^2|y_i|^2.$$

These arise from the $|\partial_i K\ W|^2 = |y_i|^2|W|^2$ terms in the potential. For the simple Kahler potential:

$$m_o^2 = 2\sqrt{3}m_{3/2}^2 \quad A = (3-\sqrt{3})m_{3/2}.$$

Also find supersymmetry-violating quadratic and cubic terms in the potential in presence of a superpotential $W(y)$; these have the form:

$$B_{ij}m_{3/2}\phi_i\phi_j + A_{ijk}m_{3/2}\phi_i\phi_j\phi_k.$$

E.g. if W is homogeneous, and of degree three:

$$e^K \frac{\partial W}{\partial y_a}\frac{\partial K}{\partial y_a^*}W + \text{c.c.} = 3m_{3/2}.$$

Additional contributions arise from

$$\left\langle \frac{\partial W}{\partial z_i} \right\rangle y_j^* W^* + \text{c.c.}.$$

In the exercises, some specific models.

Exercises

1. Review the handout on two component spinors, if the notation of Wess and Bagger (with dotted and undotted indices is familiar).

2. Verify the commutators of the Q's and the D's written as differential operators in superspace. Verify the action of the exponentiated supercharges on superfields.

3. Check that with the definition of y, Φ is chiral. Show that any function of chiral fields is a chiral field. Work out the expansion of a chiral field in powers of $\theta, \bar{\theta}$.

4. Verify that W_α transforms as in the adjoint representation, and that $\mathrm{Tr} W_\alpha^2$ is gauge invariant.

5. Derive the expression for the component lagrangian including gauge interactions and the superpotential by doing the superspace integrals.

6. Derive the supersymmetry current for a theory with several chiral fields. For a single field, Φ, and $W = 1/2\ m\Phi^2$, verify, using the canonical commutation relations, that the Q's obey the supersymmetry algebra. Work out the supercurrent for a pure supersymmetric gauge theory.

7. Work out out the spectrum of the O'Raifeartaigh model. Show that the spectrum is not supersymmetric, but verify the sum rule, $\sum(-1)^F \ m^2 = 0$.

8. Work out the spectrum of a model with a Fayet-Iliopoulos D-term and supersymmetry breaking. Again verify the sum rule.

9. Check equations for the minimum of the potential of the Polonyi model