Problem Set for lectures "Integrability in AdS/CFT", PiTP 2006, Niklas Beisert

Note: It is not necessary to solve all parts of the problems. Do as much as you can and enjoy; latter parts tend to be more challenging.

I. Spinning Strings

Introduction: Bosonic strings on $AdS_5 \times S^5$ can be expressed using the fields $\vec{X}(\sigma,\tau) \in \mathbb{R}^6$ and $\vec{Y}(\sigma,\tau) \in \mathbb{R}^{2,4}$ with $\vec{X}^2 = \vec{Y}^2 = 1$. The equations of motion read

$$\vec{X}'' - \ddot{\vec{X}} = \vec{X}\big(\vec{X}\cdot\vec{X}'' - \vec{X}\cdot\ddot{\vec{X}}\big), \qquad \vec{Y}'' - \ddot{\vec{Y}} = \vec{Y}\big(\vec{Y}\cdot\vec{Y}'' - \vec{Y}\cdot\ddot{\vec{Y}}\big)$$

and the Virasoro constraints are given by

$$\vec{X}'\cdot\dot{\vec{X}} = \vec{Y}'\cdot\dot{\vec{Y}}, \qquad (\vec{X}')^2 + (\dot{\vec{X}})^2 = (\vec{Y}')^2 + (\dot{\vec{Y}})^2.$$

The spins for rotation in the a,b-plane of $\mathbb{R}^6$ and $\mathbb{R}^{2,4}$ are given by

$$J_{ab} = \frac{\sqrt{\lambda}}{2\pi}\oint \big(X_a\dot{X}_b - X_b\dot{X}_a\big)\,d\sigma, \qquad S_{ab} = \frac{\sqrt{\lambda}}{2\pi}\oint \big(Y_a\dot{Y}_b - Y_b\dot{Y}_a\big)\,d\sigma.$$

Problem: Derive the energy and spin a spinning string using the following ansatz on $\mathbb{R}\times S^3$

$$\vec{X}(\sigma,\tau) = \begin{pmatrix} \cos\psi(\sigma)\,\cos(\omega_1\tau) \\ \cos\psi(\sigma)\,\sin(\omega_1\tau) \\ \sin\psi(\sigma)\,\cos(\omega_2\tau) \\ \sin\psi(\sigma)\,\sin(\omega_2\tau) \\ 0 \\ 0 \end{pmatrix} \qquad \vec{Y}(\sigma,\tau) = \begin{pmatrix} \cos\varepsilon\tau \\ \sin\varepsilon\tau \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

a) Substitute the ansatz into the equations of motion and Virasoro contraints to obtain differential equations for the profile $\psi(\sigma)$. Show that the resulting equations are compatible, i.e. that the Virasoro constraints equal the integrated equations of motion.

b) Solve the equations of motion in the special case $\omega_1 = \omega_2$. Restrict to periodic solutions with $\psi(2\pi) = \psi(0) + 2\pi m$.

c) Find the spins J_{12}, J_{34} and energy $E = S_{12}$ of the solution. Express the energy as a function of the spins.

d) If you can, repeat b) and c) for the more general case $\omega_1 \neq \omega_2$ and also for folded strings with $\psi(2\pi) = \psi(0)$.

II. Spectral Curve

Introduction: The spectral curve $p'(x)$ for the $\mathbb{R}\times S^3$ sector of string theory has the following properties: The A-cycle of a cut encircles it. The B-cycle of a cut starts and ends at $x=\infty$ and passes through the cut. The A-cycle is zero and the B-cycles defines the mode number of the cut

$$\oint_{\mathcal{A}_k} dp(x) = 0 \quad \text{and} \quad \oint_{\mathcal{B}_k} dp(x) = 2\pi n_k.$$

The winding number m is defined through the integral

$$\int_0^\infty dp(x) = -2\pi m.$$

The spins J_{12} and J_{34} can be read off from the expansion at $x=0$ and $x=\infty$

$$p(x) = 2\pi m - \frac{2\pi(J_{12}+J_{34})}{\sqrt{\lambda}}\,x + \mathcal{O}(x^2) \quad \text{and} \quad p(x) = \frac{2\pi(J_{12}-J_{34})}{\sqrt{\lambda}}\,\frac{1}{x} + \mathcal{O}(x^{-2})$$

and the energy E from the expansion at $x = \pm 1$

$$p = \frac{\pi E}{\sqrt{\lambda}} \frac{1}{x \mp 1} + \mathcal{O}\big((x \mp 1)^0\big).$$

Problem: Use the single-cut ansatz

$$p'(x) = \frac{cx^3 + dx^2 + ex + f}{(x^2-1)^2\sqrt{x^2+ax+b}}$$

to solve for the energy of a solution with $J_{12} = J_{34} = \frac{1}{2}J$ and winding number m.

a) Use the above relations to constrain all the coefficients a, b, c, d, e, f *before* integrating $p'(x)$.

b) Use the relation of the winding number m to obtain the energy as a function of J, m, λ. Compare the answer to the solution to problem I.

c) What is the mode number n associated to the cut?

III. Heisenberg Chain

Introduction: Consider the periodic Heisenberg spin chain with K up spins and $L - K$ down spins, i.e. L sites in total. The Heisenberg Hamiltonian is given by

$$\mathcal{H} = \sum_{a=1}^{L} \big(\mathcal{I}_{a,a+1} - \mathcal{P}_{a,a+1}\big).$$

Here $\mathcal{I}_{a,a+1}, \mathcal{P}_{a,a+1}$ are the identity and permutation operators acting on two adjacent sites a and $a+1$ (periodically identified: $L + 1 \equiv 1$).

Problem: Compute the spectrum of $\mathcal{H}$ in the cases specified below: First, enumerate all states. Second, act with $\mathcal{H}$ on these states and thus represent it as a matrix in this basis. Finally, find the eigenvalues of this matrix.

a) Compute the spectrum for the states $L = 3$ and arbitrary number of spin flips K. How do these fit into multiplets of SU(2)?

From now on restrict to cyclic states, i.e. identify all states which are equivalent under cyclic permutations.

b) Compute the spectrum for the states $L = 4$, $K = 2$ and $L = 6$, $K = 2, 3$.

c) Compute the spectrum for the states with $K = 2$ and arbitrary length L.

VI. Bethe Equations

Introduction: The Bethe equations for the Heisenberg chain are

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^{K} \frac{u_k - u_j + i}{u_k - u_j - i}, \qquad \text{for } k = 1, \ldots, K.$$

For each solution of these equations (with distinct u_k) there exist an eigenstate of the Heisenberg Hamiltonian with energy and (exponentiated) momentum

$$E = \sum_{k=1}^{K} \left(\frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}} \right) \quad \text{and} \quad e^{iP} = \prod_{k=1}^{K} \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}.$$

Problem: Use the Bethe equations to derive the energies of states.

a)-c) Repeat parts a) to c) of Problem III and compare the energies. Note: $u_k = \infty$ is allowed, even with multiplicity. Cyclic states have zero momentum. The state $L = 6, K = 3$ is singular, can you find it?

d) Use the BDS Bethe equations to find higher-loop corrections to scaling dimensions: Replace every instance of $u_k \pm \frac{i}{2}$ (on LHS of BE, in energy & momentum) by $x(u_k \pm \frac{i}{2})$ where $x(u) = \frac{1}{2}u + \frac{1}{2}u\sqrt{1 - 2g^2/u^2}$. Expand the equations order by order in g^2.