Lecture III: Integrability in Large-N Gauge Theories

- The Heisenberg $XXX_{1/2}$ Spin Chain
- Bethe Ansatz
  - Ferromagnetic Vacuum
  - Spin Flips as Particles
  - Asymptotic Regions
  - Scattering Phase
  - Factorised Scattering
  - Bethe Equations
- Heisenberg XXX_s Model
- Bethe Ansätze with Flavour, Yang-Baxter Relation
- Nested Bethe Ansatz & Generic Algebras

The Heisenberg $XXX_{1/2}$ Spin Chain & Gauge Theory

$$\operatorname{Tr}\,\Phi_1\Phi_2\Phi_1\Phi_1\Phi_1\Phi_2\Phi_2\Phi_1\Phi_1 \longleftrightarrow |\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow\uparrow\downarrow\downarrow\rangle$$

local operator — spin chain state

$SU(2) \subset SO(6)$ symmetry — $\uparrow,\downarrow$ transform as spin $\frac{1}{2}$ of $SU(2)$

Classical Dimension: L (length of chain)

One-Loop Dimension: $g^2 E$ (eigenvalue of Hamiltonian H)

$$H = \sum_{k=1}^{L} H_{k,k+1} \qquad H_{k,k+1} = \underbrace{I_{k,k+1}}_{\text{Identity}} - \underbrace{P_{k,k+1}}_{\text{Permutation of two spins}} = \frac{1}{2}\left(1 - \vec{\sigma}_k \cdot \vec{\sigma}_{k+1}\right)$$

Same coupling for X, Y & Z
$\Rightarrow$ "XXX"
$\Rightarrow$ $SU(2)$ inv.

What is the spectrum of the linear operator H?

Brute Force

- list all states with given $n_\uparrow, n_\downarrow$
- evaluate H in this basis: $\binom{n_\uparrow}{L} \times \binom{n_\uparrow}{L}$ matrix
- diagonalise H_{mat}

Straightforward, but hard when basis large (~ 10000, $L=20$)

Bethe Ansatz: Vacuum and Spin Flip

H. Bethe '31

Ferromagnetic Vacuum $|0\rangle = |\downarrow\downarrow\downarrow\downarrow\downarrow\dots\downarrow\downarrow\rangle$

Find $H_{12}|\downarrow\downarrow\rangle = |\overset{I}{\downarrow\downarrow}\rangle - |\overset{P}{\downarrow\downarrow}\rangle = 0$

$\Rightarrow H|0\rangle = 0$

Position k

Spin Flip $|k\rangle = |\downarrow\dots\downarrow\uparrow\downarrow\dots\dots\downarrow\rangle$

Hamiltonian homogeneous $\Rightarrow$ Eigenstates are plane waves

Momentum Eigenstate (infinite chain for time being)

$$|p\rangle = \sum_{k=-\infty}^{+\infty} e^{ipk}|k\rangle$$

Act with H to find eigenvalue

$$H|p\rangle = \sum_{k=-\infty}^{+\infty} e^{ipk}\Big(\overbrace{|k\rangle - |k-1\rangle}^{H_{k-1,k}} + \overbrace{|k\rangle - |k+1\rangle}^{H_{k,k+1}}\Big)$$

shift $k \to k\pm1$

$$= \sum_{k=-\infty}^{+\infty} e^{ipk}\left(1 - e^{ip} + 1 - e^{-ip}\right)|k\rangle$$

$$= 2(1-\cos p)\,|p\rangle$$

$$= 4\sin^2 \tfrac{p}{2}\,|p\rangle$$

$$= e(p)\,|p\rangle$$

with Dispersion Relation $e(p) = 4\sin^2 \tfrac{p}{2} = 2 - 2\cos p$

Two particle States & Scattering Phase

Position Eigenstate

$$|k<l\rangle = |\downarrow \dots \downarrow \overset{k}{\uparrow} \downarrow \dots \downarrow \overset{l}{\uparrow} \downarrow \dots \downarrow\rangle$$

Asymptotic Momentum Eigenstate

$$|p<q\rangle = \sum_{k<l=-\infty}^{+\infty} e^{ipk+iql} \, |k<l\rangle$$

Almost Eigenstate with Eigenvalue $e(p)+e(q)$:

When Spin Flips are far enough apart, they do not see each other.

Act with $H-e(p)-e(q)$

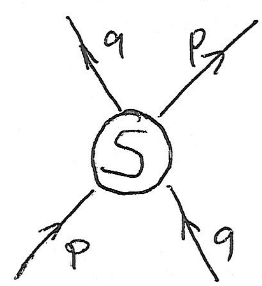
contact term

$$(H-e(p)-e(q))\,|p<q\rangle = \sum_{k=-\infty}^{+\infty} e^{i(p+q)k}\left(e^{ip+iq}-2e^{iq}+1\right)|k<k+1\rangle$$

Similarly (exchange particles)

$$(H-e(p)-e(q))\,|q<p\rangle = \sum_{k=-\infty}^{+\infty} e^{i(p+q)k}\left(e^{ip+iq}-2e^{+ip}+1\right)|k<k+1\rangle$$

Construct Exact Eigenstate



$$|p,q\rangle = |p<q\rangle + S\,|q<p\rangle$$

with Scattering Phase

$$S(p,q) = -\frac{e^{ip+iq}-2e^{iq}+1}{e^{ip+iq}-2e^{ip}+1} = e^{2i\varphi(p,q)}$$

Exact Eigenstate

$$H|p,q\rangle = \left(e(p)+e(q)\right)|p,q\rangle$$

$6 = 3!$ asymptotic regions for 3 particles

Match up regions at contact terms. Find Eigenstate

$$|p_1, p_2, p_3\rangle = |p_1 < p_2 < p_3\rangle + S_{12}\,|p_2 < p_1 < p_3\rangle + S_{23}\,|p_1 < p_3 < p_2\rangle$$
$$+ S_{13} S_{12}\,|p_2 < p_3 < p_1\rangle + S_{13} S_{23}\,|p_3 < p_1 < p_2\rangle + S_{12} S_{13} S_{23}\,|p_3 < p_2 < p_1\rangle$$

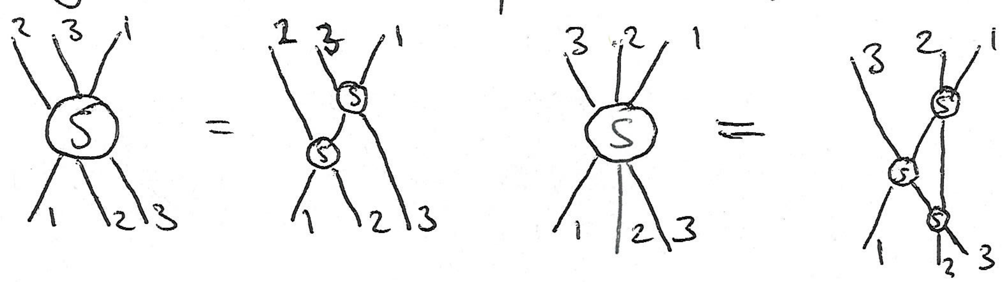
with eigenvalue $e(p_1) + e(p_2) + e(p_3)$:

$$H\,|p_1, p_2, p_3\rangle = \big(e(p_1) + e(p_2) + e(p_3)\big)\,|p_1, p_2, p_3\rangle$$

Integrability :

Scattering Factorises for any number of particles   Bethe '31

eg.



Only need Two-Particle Scattering Phase to construct any Eigenstate on infinite chain.

Periodicity & Bethe Equations

We have: Infinite Chain

We want: Finite, periodic chain

→ same as periodic states on infinite chain.

Move one excitation p_k past forward or backward

- L sites of the chain : e^{+iLp_k} | e^{-iLp_k}
- k-1 other particles : $\times \prod_{\substack{j=1\\j\neq k}}^{K} S_{kj}$ | $\times \prod_{\substack{j=1\\j\neq k}}^{K} S_{jk}$

Should end up with the same state : $=1$

Bethe Equations (trigonometric)

$$1 = e^{-ip_k L} \prod_{\substack{j=1\\j\neq k}}^{K} \left(- \frac{e^{ip_k + ip_j} - 2e^{ip_k} + 1}{e^{ip_k + ip_k} - 2e^{ip_j} + 1} \right) \quad \text{for all } k=1..K$$

Reparameterise $p_k = 2\,\mathrm{arccot}\, 2u_k$ u_k rapidity $u_k = \frac{1}{2}\cot\frac{p_k}{2}$

Bethe Equations (algebraic)

$$1 = \left(\frac{u_k - \frac{i}{2}}{u + \frac{i}{2}} \right)^L \prod_{\substack{j=1\\j\neq k}}^{K} \frac{u_k - u_j + i}{u_k - u_j - i}$$

Total Energy $E = \sum_{j=1}^{K} e(p_j) = \sum_{j=1}^{K} 4\sin^2 \frac{p_j}{2} = \sum_{j=1}^{K} \left(\frac{i}{u_j + \frac{i}{2}} - \frac{i}{u_j - \frac{i}{2}} \right)$

Total Momentum $e^{iP} = \prod_{j=1}^{K} e^{ip_j} = \prod_{j=1}^{K} \frac{u_j + i/2}{u_j - i/2}$

Cyclicity of trace: $\mathrm{Tr}\, ABCD\ldots XYZ = \mathrm{Tr}\, BCD \ldots XYZA$

requires $e^{iP} = 1$ zero momentum condition.

Other Bethe Ansätze

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Bethe Ansatz works also for long-range systems to some extent.

Sutherland '78
Staudacher '04

- Scattering Phase to tie up asymptotic regions.
- Contact terms more involved (long-ranged)

For gauge theory SU(2) sector Hamiltonian → Lecture I.

Bethe Ansatz for Heisenberg XXX_1 Chain (Spin 1)

3 States, say $|0\rangle$ $|1\rangle$ $|2\rangle$ Mixing:

$H_{12}|00\rangle \to |00\rangle$

$H_{12}|01\rangle, |10\rangle \to |01\rangle, |10\rangle$

$H_{12}|02\rangle, |11\rangle, |20\rangle \to |02\rangle, |11\rangle, |20\rangle$

$H_{12}|12\rangle, |21\rangle \to |12\rangle, |21\rangle$

$H_{12}|22\rangle \to |22\rangle$

$|0\rangle$ Spin vacuum

$|1\rangle$ one excitation

$|2\rangle$ double excitation on single site

b/c in $|0\rangle$ vac: $|..020...\rangle \to |...110...\rangle$

$|2\rangle$ disintegrates into $|1\rangle + |1\rangle$

Bethe Ansatz ... Bethe Equations

BE $\left(\frac{u_k - i}{u_k + i}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{k} \frac{u_k - u_j + i}{u_k - u_j - i}$

Mom $e^{iP} = \prod_{j=1}^{k} \frac{u_j + i}{u_j - i}$

Eng $E = \sum_{j=1}^{k} \left(\frac{i}{u_j + i} - \frac{i}{u_j - i}\right)$

spins for XXX_s: $\left(\frac{u - is}{u + is}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{k} \frac{u_k - u_j + i}{u_k - u_j - i}$

$e^{iP} = \prod_{j=1}^{k} \left(\frac{u_j + is}{u_j - is}\right)$

$E = \sum_{j=1}^{k} \left(\frac{i}{u_j + is} - \frac{i}{u_j - is}\right).$

~~Nested~~ Bethe Ansatze with Flavour

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For $SU(N)$ chain with fundamental spins

$|1\rangle, |2\rangle, \ldots, |N\rangle$

$$H_{12} = I_{12} - P_{12} \quad \text{ie. } H_{12}|ab\rangle = |ab\rangle - |ba\rangle$$

Vacuum: $|0\rangle = |\ldots 1111 \ldots\rangle \qquad E = 0$

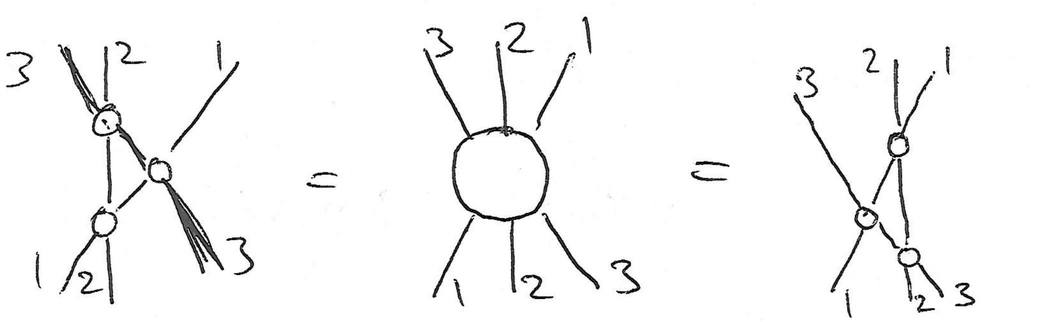
One Particle states: Flavour $a = 2, \ldots, N$ $\quad N-1$ flavours

$$|(p,a)\rangle = \sum_k e^{ipk} |\ldots 111 \overset{k}{a} 111 \ldots\rangle$$

Scattering Phase becomes Scattering Matrix

$$|(p,a),(q,b)\rangle = |(p,a) < (q,b)\rangle + \sum_{cd} S_{ab}^{cd} |(q,d) < (p,c)\rangle$$

For Integrable Systems: S must obey Yang-Baxter-Relation



$$S_{23} S_{13} S_{12} = S_{123}^{321} = S_{12} S_{13} S_{23}$$

S-matrix changes flavour of scattering particles.
Cannot use (easily) for Bethe Equations yet.

Define new vacuum of type-2 excitations only

$$|0\rangle_{\mathrm{II}} = |2_1 2_2 2_3 2_4 \ldots\rangle \equiv |(p_1,2),(p_2,2),(p_3,2),\ldots\rangle$$

S-matrix applied easily to $|0\rangle_{\mathrm{II}}$ because $22 \to 22$ (no flavour)

Introduce new particle with suitable (inhomogeneous) wave function

$$|(a,p)\rangle_{\mathrm{II}} = \sum_k \psi_k(p)\, |2_1 2_2 2_3 \ldots a_k \ldots\rangle$$

Particle can take flavours $a = 3, \ldots, N$

2-particle states lead to new S-matrix S_{II} with flavours $a = 3, \ldots, N$

	Before this step	After this step
excitations:	vacuum 1	vacuum 1
	excitations $1 \to 2$	new vacuum $1 \to 2$
	$1 \to 3$	excitations $2 \to 3$
	$\vdots$	$\vdots$
	$1 \to N$	$2 \to N$

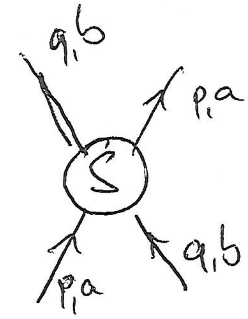
Iterate this step until all flavours are gone (nesting)

Final set of particles flavours:

vacuum 1

excitations: $1 \to 2,\ 2 \to 3,\ \ldots,\ N-1 \to N$

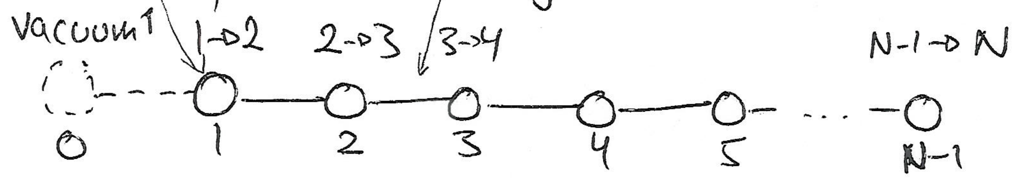
After Nesting S-matrix is diagonal:



Elements of S-matrix:

$$S_{kk} = \frac{u-v+i}{u-v-i} \qquad S_{k,k+1} = \frac{u-v-\frac{i}{2}}{u-v+\frac{i}{2}} \qquad \text{otherwise } S_{kl} = 1$$

Described by Dynkin diagram of SU(N)



Cartan Matrix:

$$M_{kl} = \begin{pmatrix} +2 & -1 & & & & \\ -1 & +2 & -1 & & & \\ & -1 & +2 & -1 & & \\ & & -1 & \ddots & -1 & \\ & & & -1 & +2 & -1 \\ & & & & -1 & +2 \end{pmatrix}$$

$$S_{kl} = \frac{u-v+\frac{i}{2}M_{kl}}{u-v-\frac{i}{2}M_{kl}}$$

for standard nearest-neighbour models (one-loop gauge)

also works for superalgebras eg. PSU(2,2|4)

Coupling to Spin Vacuum

for standard NN-models (one-loop gauge)

$$S_{k0} = \frac{u+\frac{i}{2}V_k}{u-\frac{i}{2}V_k}$$

Momentum

V_k are Dynkin labels of Spin representation

Energy:

$$e_k(u) = \frac{i}{u+\frac{i}{2}V_k} - \frac{i}{u-\frac{i}{2}V_k}$$

eg. $V_k = (1, 0, 0, \ldots, 0)$ for SU(N) fundamental.

Bethe Equations as before, move one ex past all others around chain.