QCD at Colliders
Lecture 3

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The Drell-Yan process

**LO partonic cross section:**

\[ \hat{s} = x_1 x_2 s = M_{e^+e^-}^2 \]

\[ \hat{\sigma}(q\bar{q} \to e^+e^-) = \frac{1}{2 \hat{s}} \frac{1}{4 N_c^2} \sum_{h,c} |A_4|^2 \]

\[ = \frac{4 \pi \alpha^2}{3} \frac{1}{N_c} Q_q^2 \]

\[ \frac{d\hat{\sigma}}{dM^2} = \frac{\sigma_0}{N_c} Q_q^2 \delta(\hat{s} - M^2), \quad \sigma_0 = \frac{4 \pi \alpha^2}{3 M^2} \]

**LO hadronic cross section:**

\[ \frac{d\sigma}{dM^2} = \int_0^1 dx_1 dx_2 \sum_q [q(x_1)\bar{q}(x_2) + \bar{q}(x_1)q(x_2)] \frac{d\hat{\sigma}}{dM^2} \]

\[ = \frac{\sigma_0}{N_c} \int_0^1 dx_1 dx_2 \delta(x_1 x_2 s - M^2) \sum_q Q_q^2 [q(x_1)\bar{q}(x_2) + \bar{q}(x_1)q(x_2)] \]

\[ = \frac{\sigma_0 s}{N_c} \int_0^1 dx_1 dx_2 \delta(x_1 x_2 - \tau) \sum_q Q_q^2 [q(x_1)\bar{q}(x_2) + \bar{q}(x_1)q(x_2)], \quad \tau \equiv \frac{M^2}{s} \]

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QCD at Colliders: Lect. 3
Drell-Yan rapidity distribution

\[ Y = \frac{1}{2} \ln \left( \frac{E + p_z}{E - p_z} \right) \]

\[ \exp(2Y) = \frac{E + p_z}{E - p_z} = \frac{P_2 \cdot P_Z}{P_1 \cdot P_Z} = \frac{1}{x_2} \frac{p \bar{q} \cdot P_Z}{x_1 p q \cdot P_Z} = \frac{x_1}{x_2} \]

Combined with mass measurement,

\[ x_1 x_2 = \tau = \frac{M^2}{s} \]

Double distribution

\[ \frac{d^2\sigma}{dM^2 dY} = \frac{\sigma_0}{N_c s} \sum_q \sigma_q^2 \left[ q(x_1)\bar{q}(x_2) + \bar{q}(x_1)q(x_2) \right] \]

Measures product of quark and antiquark distributions at

\[ x_1 = \sqrt{\tau e^Y}, \quad x_2 = \sqrt{\tau e^{-Y}} \]
NLO QCD corrections to Drell-Yan production

\[ |A_5|^2 = \frac{s_{13}^2 + s_{15}^2 + s_{23}^2 + s_{25}^2}{s_{12}s_{34}s_{45}} \]

As at LO, average over decay direction of $e^+$ and $e^-$:

\[
\langle k_1^\mu k_1^\nu \rangle_\Omega \equiv \int \frac{d\Omega_{e^+ e^-} k_1^\mu k_1^\nu}{4\pi} = -\frac{s_{12}}{12} \eta^{\mu \nu} + \frac{1}{3} (k_1 + k_2)^\mu (k_1 + k_2)^\nu = \langle k_2^\mu k_2^\nu \rangle_\Omega
\]

\[
\langle s_{13}^2 \rangle_\Omega = \langle s_{23}^2 \rangle_\Omega = \frac{1}{3} (s_{13} + s_{23})^2 = \frac{1}{3} (s_{34} + s_{35})^2
\]

\[
\Rightarrow \langle |A_5|^2 \rangle_\Omega = \frac{2}{3} \frac{(s_{34} + s_{35})^2 + (s_{35} + s_{45})^2}{s_{12}s_{34}s_{45}}
\]
Phase space for DY @ NLO

Could use gluon energy, angle in CM frame, $E_4, \theta$

Trade for $z, y \in [0,1]$ defined by:

$z = \frac{s_{12}}{s_{35}}$

$y = \frac{1 - \cos \theta}{2}$

$E_4 = \frac{s_{34} + s_{45}}{2 \sqrt{s_{35}}} = \frac{s_{12} - s_{35}}{2 \sqrt{s_{35}}} = \frac{1 - z}{2} \sqrt{s_{35}}$

$s_{34} = -\sqrt{s_{35}} E_4 (1 - \cos \theta) = -y(1 - z) s_{35}$

$s_{45} = -\sqrt{s_{35}} E_4 (1 - \cos \theta) = -(1 - y)(1 - z) s_{35}$

$s_{12} = M^2 = z s_{35}$

Cross section:

$\langle |A_5|^2 \rangle_\Omega = \frac{2}{3 M^2} \frac{(1 - y(1 - z))^2 + (1 - (1 - y)(1 - z))^2}{y(1 - y)(1 - z)^2}$

$\propto \left( \frac{\mu^2}{s_{35}} \right) \epsilon \frac{d^3-2\epsilon}{2E_4} \propto \left( \frac{\mu^2}{M^2} \right) \epsilon \frac{dE_4}{E_4} E_4^{1-2\epsilon} \frac{d\cos \theta (\sin^2 \theta)^{-\epsilon}}{d\Omega^{1-2\epsilon}}$

$\propto \left( \frac{\mu^2}{M^2} \right) \epsilon \frac{dy \, dz \, [y(1 - y)]^{-\epsilon} \, z^\epsilon \, (1 - z)^{1-2\epsilon}}{d\Omega^{1-2\epsilon}}$
QCD corrections to DY (cont.)

Integral to do:

\[ I = \left( \frac{\mu^2}{M^2} \right)^\epsilon z^\epsilon (1 - z)^{-1-2\epsilon} \times \int_0^1 dy \frac{[y(1 - y)]^{-\epsilon(1 - y(1 - z))^2 + (1 - (1 - y)(1 - z))^2}}{y(1 - y)} \]

Hard collinear divergences are at \( y = 0, 1 \)

Separate using

\[ \frac{1}{y(1 - y)} = \frac{1}{y} + \frac{1}{1 - y} \]

Expand \( 1/y \) term in cross section about \( y = 0 \)

\[ I = 2 \left( \frac{\mu^2}{M^2} \right)^\epsilon z^\epsilon (1 - z)^{-1-2\epsilon} \int_0^1 dy \frac{y^{-1-\epsilon}}{1 + z^2 - 2y(1 - y)(1 - z)^2} \times (1 - \epsilon \ln(1 - y)) \]

\[ = 2 \left( \frac{\mu^2}{M^2} \right)^\epsilon z^\epsilon (1 - z)^{-1-2\epsilon} \left[ \frac{1 + z^2}{\epsilon} - (1 - z)^2 + O(\epsilon) \right] \]
Including a few other omitted prefactors:

\[
d\tilde{\sigma}^{\text{NLO, real}} \frac{\text{d}}{\text{d}M^2} = \frac{\sigma_0}{N_c} Q^2 q \frac{\alpha_s}{2\pi} C_F \left[ 2 \left( -\frac{1}{\epsilon} - \ln(4\pi) + \gamma \right) \frac{1 + z^2}{1 - z} - 2 \frac{1 + z^2}{1 - z} \left( -2 \ln(1 - z) + \ln z - \ln \frac{M^2}{\mu^2} \right) - 2(1 - z)^2 \right]
\]

divergence absorbed into \( q(x) \) in MS factorization scheme

correction to cross section

\[
q(x, \mu) = q_0(x) + \frac{\alpha_s(\mu)}{2\pi} \left( -\frac{1}{\epsilon} - \ln(4\pi) + \gamma_E \right) \int_x^1 \frac{d\xi}{\xi} \left[ P_{qq}^{(0)}(x/\xi)q_0(\xi) + P_{qq}^{(0)}(x/\xi)g_0(\xi) \right] \\
= q_0(x) + \frac{\alpha_s(\mu)}{2\pi} \left( -\frac{1}{\epsilon} - \ln(4\pi) + \gamma_E \right) \int_x^1 \frac{dz}{z} \left[ C_F \frac{1 + z^2}{1 - z} q_0(x/z) + P_{qq}^{(0)}(z)g_0(x/z) \right]
\]
Finally, virtual graph has support only at $z=1$. -- kinematics same as at LO. Regulates $1/(1-z)$ into plus distribution. Final result:

$$
\frac{d\sigma^{\text{NLO}}}{dM^2} = \frac{\sigma_0}{N_c s} \int_0^1 dx_1 dx_2 dz \delta(x_1 x_2 z - \tau) \sum_q Q_q^2 \left[ q(x_1, \mu_F) \bar{q}(x_2, \mu_F) \left( \delta(1-z) + \frac{\alpha_s(\mu_R)}{2\pi} C_F D_q(z, \mu_F) \right) 
+ g(x_1, \mu_F) \left( q(x_2, \mu_F) + \bar{q}(x_2, \mu_F) \right) \frac{\alpha_s(\mu_R)}{2\pi} T_R D_g(z, \mu_F) 
+ (x_1 \leftrightarrow x_2) \right]
$$

where

$$
D_q(z, \mu_F) = 4(1 + z^2) \left( \frac{\ln(1-z) + \ln(M/\mu_F)}{1-z} \right) + 
-2 \frac{1 + z^2}{1-z} \ln z + \delta(1-z) \left( \frac{2}{3} \pi^2 - 8 \right)
$$

singular distribution as $z \to 1$
QCD corrections to DY (cont.)

and

\[ D_g(z, \mu_F) = (z^2 + (1 - z)^2) \left[ \ln \frac{(1 - z)^2}{z} + 2 \ln \frac{M}{\mu_F} \right] + \frac{1}{2} + 3z - \frac{7}{2}z^2 \]

comes from the \( qg \to q\gamma^* \) subprocess:

- Cross section related by crossing to \( \bar{q}q \to g\gamma^* \)
- Remove \( g \to qq \) collinear singularity in same way
- Note that there is no \( 1/(1-z) \) (soft gluon) singularity in this term.
Why are NLO corrections large?

+ 30% typical for quark-initiated ($W/Z, \ldots$)

+ 80-100% for some gluon-initiated ($gg \rightarrow \text{Higgs} + X$)

This is much bigger than $R_{e^+e^-} = 1 + \frac{\alpha_s}{\pi} \approx 1 + \frac{0.1}{\pi} \approx 1 + 0.03$ !!
Some answers (not all for all processes)

1. LO parton distribution fits not very reliable due to large theory uncertainties

2. New processes can open up at NLO. In W/Z production at Tevatron or LHC, $qg \rightarrow \gamma^* q$ opens up, and $g(x)$ is very large – but correction is negative!

3. Large $\pi^2$ from analytic continuation from space-like region where pdfs are measured (DIS) to time-like region (Drell-Yan/W/Z):

$$2 \text{ Re} \left[ 1 + \frac{\alpha_s}{\pi} C_F \left(-\frac{1}{\epsilon^2}\right) \Re \left[ \left( \frac{\mu^2}{-Q^2} \right)^\epsilon - \left( \frac{\mu^2}{+Q^2} \right)^\epsilon \right] \right]$$

$$= 1 + \frac{\alpha_s}{\pi} C_F \left(-\frac{1}{\epsilon^2}\right) \Re \left[ \exp(i\pi \epsilon) - 1 \right] = 1 + \frac{\alpha_s}{\pi} C_F \frac{\pi^2}{2}$$
4. Soft-gluon/Sudakov resummation

• A prevalent theme in QCD whenever one is at an edge of phase space.
• Infrared-safe but sensitive to a second, smaller scale
• Same physics as in (high-energy) QED:
  • What is prob. of no $\gamma$ with $E > \Delta E$, $\theta > \Delta \theta$?

$$P = 1 - \frac{\alpha}{\pi} \int_{\Delta E} \frac{dE}{E} \int_{\Delta \theta} \frac{d\theta}{\theta} + \cdots = 1 - \frac{\alpha}{\pi} \ln \Delta E \ln \Delta \theta + \cdots$$

$$= \exp \left( -\frac{\alpha}{\pi} \ln \Delta E \ln \Delta \theta \right) + \cdots$$

leading double logarithms -- in contrast to single logs of renormalization group, DGLAP equations.

exponentiation because soft emissions are independent
Example: $e^+e^- \text{ Thrust } T \to 1$

Hard, wide angle radiation forbidden

$E_{\text{min}}(T) \propto (1 - T)\sqrt{s} \ll \sqrt{s}$

$$\frac{1}{\sigma} \frac{d\sigma}{dT} \approx \frac{4C_F \alpha_s}{\pi} \int \frac{dE \, d\theta}{E \, \theta} \delta \left(1 - T - \frac{E \theta^2}{\sqrt{s}}\right)$$

$$= \frac{2C_F \alpha_s}{\pi} \left(\frac{1}{1 - T} \int_{E_{\text{min}}(T)}^{E_{\text{max}}} \frac{dE}{E}\right)$$

$$= -\frac{2C_F \alpha_s}{\pi} \ln(1 - T)$$

$$P(1 - T < \tau) = 1 - \frac{C_F \alpha_s}{\pi} \ln^2 \tau + \cdots$$

$$\approx \exp \left( -\frac{C_F \alpha_s}{\pi} \ln^2 \tau \right)$$

Known how to resum $\ln \tau$'s in exponent to NLL (next-to-leading-log) accuracy for many variables, NNLL for some.
**e^+e^- jets with \( y_{\text{cut}} \rightarrow 0 \)**

\[
y_{\text{cut}} = \frac{2E_i E_j (1 - \cos \theta_{ij})}{s} \approx \frac{E \theta^2}{\sqrt{s}}
\]

Hard, wide angle radiation again forbidden -- "pencil-thin jets"

Two-jet rate exponentiates like thrust.

Higher multijet rates can be resummed, but not exponentiated. \( \sim \alpha_s^n L^{2n} \)

Like thrust, dramatic effects at Z pole only start to happen when physical scale is getting close to \( \Lambda_{\text{QCD}} \)

-- large hadronization corrections
Hadron collider examples

\( p_T(Z) \), important application to \( p_T(W) \),
\( m_W \) measurement at Tevatron was discussed by Dieter

Another class of examples is provided by production of heavy states, like
- top quark at the Tevatron (\( W \) and \( Z \) production less so),
- even a light Higgs boson at the LHC, via \( gg \rightarrow H \)

Called threshold resummation or limit,
where \( x = M^2/s \).
Can be important for \( x << 1 \) though.
For \( m_H = 120 \) GeV at 14 TeV LHC, \( x = 10^{-4} \)!
Radiation is being suppressed because you are running out of phase space – parton distributions are falling fast.
Threshold Resummatation

We saw the first log of this type in the NLO corrections to Drell-Yan/W/Z production:

\[ C_F D_q(z, \mu_F) = 4C_F (1 + z^2) \left( \frac{\ln(1 - z) + \ln(M/\mu_F)}{1 - z} \right) + \]

\[-2 \frac{1 + z^2}{1 - z} \ln z + \delta(1 - z) \left( \frac{2}{3} \pi^2 - 8 \right)\]

Also a double-log expansion:

\[ D_q^{(n)}(z, \mu_F) \propto (C_F \alpha_s)^n \left[ \left( \frac{\ln^{2n+1}(1 - z)}{1 - z} \right)_+ + \cdots \right] \]

For \( gg \to H \), same leading behavior at large \( z \).
Except color factor is much bigger: \( C_A = 3 \), not \( C_F = 4/3 \)

\[ D_{gg \to H}^{(n)}(z, \mu_F) \propto (C_A \alpha_s)^n \left[ \left( \frac{\ln^{2n+1}(1 - z)}{1 - z} \right)_+ + \cdots \right] \]
Fast falling pdfs -- worse for gluons

A. Because the same suppression happens in the DIS process used to measure the pdfs. Both parton distributions “bigger than you thought”: $2 - 1 > 0$.

Q: If it is called Sudakov suppression, why does it increase the cross section?

- $gg \rightarrow \text{light Higgs}$ at LHC
- $q\bar{q} \rightarrow W, Z$ at LHC
- $q\bar{q} \rightarrow W, Z$ at Tevatron

pdfs \rightarrow \text{partonic cross section}
Conclusions

- QCD at colliders is an extremely rich field.
- I was only able to scratch the surface of it here.
- Indeed, at hadron colliders, the physics is QCD – up to small, electroweak corrections!
- So, to uncover new physics of electroweak strength, we will need to understand QCD at colliders quite well.
- There is plenty of room for fresh, new ideas from young theorists and experimentalists (you!)