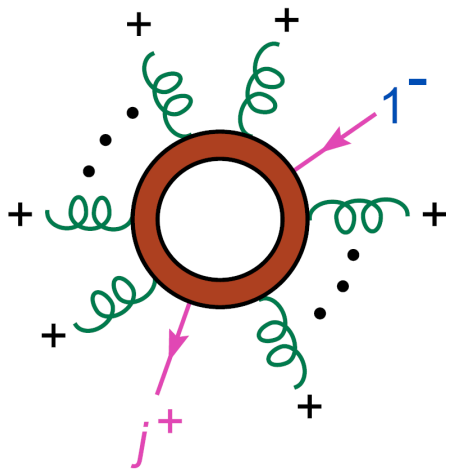
On-Shell Recursion Relations for QCD Tree & Loop Amplitudes



Lance Dixon, SLAC

Prospects in Theoretical Physics, IAS Princeton
July 21, 2005

R. Britto, F. Cachazo, B. Feng, hep-th/0412308;
R. Britto, F. Cachazo, B. Feng, E. Witten, hep-th/0501052

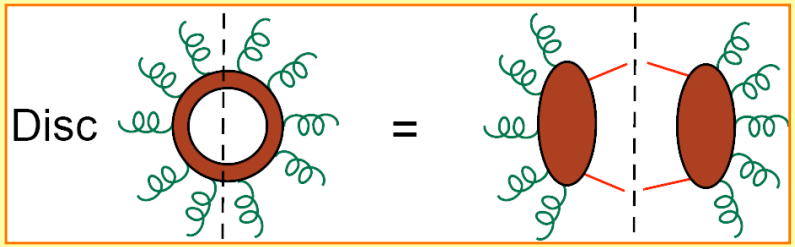
Z. Bern, LD, D. Kosower, hep-th/0501240, hep-ph/0505055, hep-ph/0507005

Motivation

- Need a flexible, efficient method to extend the range of known tree, and particularly 1-loop QCD amplitudes, for use in NLO corrections to LHC processes, etc.
- Unitarity is an efficient method for determining imaginary parts of loop amplitudes:

$$S = 1 + iA$$
$$S^\dagger S = 1 \quad \Rightarrow \quad 1 = (1 - iA^\dagger)(1 + iA)$$
$$\Rightarrow \quad -i(A - A^\dagger) = 2\,\mathrm{Im}\,A = \mathrm{Disc}\,A = A^\dagger A$$

- Efficient because it recycles trees into loops



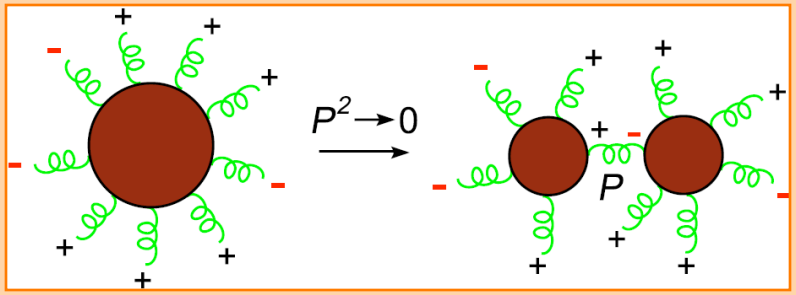
Motivation (cont.)

- But unitarity can miss rational functions that have no cut.
- These functions can be recovered (using dimensional analysis) if cuts are computed to higher-order in ε, in dim. reg. with D=4-2ε:

$$R(s_{ij}) \;\rightarrow\; R(s_{ij})(-s)^{-\varepsilon} = R(s_{ij})(1-\varepsilon\ln(-s))$$

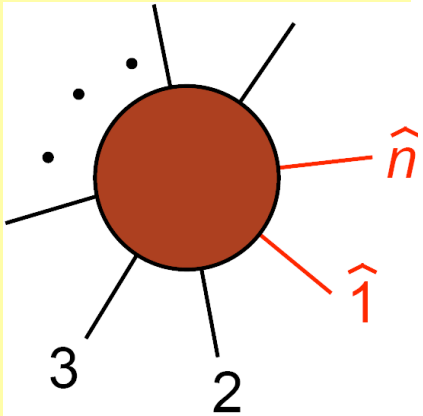
- But (4-2ε)-dimensional tree amplitudes are more complicated than 4-dimensional ones. Seems too much information is used.
- *n*-point amplitudes also factorize onto lower-point amplitudes.
- At tree-level this information has recently been systematized into on-shell recursion relations

  (BCF, BCFW (2004-5),…)
- Efficient because it recycles trees into trees
- **Can also do the same for loops**



On-shell tree recursion

- BCFW consider a family of on-shell amplitudes $A_n(z)$ depending on a complex parameter z which shifts the momenta. (twistor-inspired – but that's another story; see review by Cachazo, Svrcek (2005))
- Best described using spinor variables.
- For example, the (n,1) shift:



$$\lambda_1 \to \hat{\lambda}_1 = \lambda_1 + z\lambda_n \qquad \tilde{\lambda}_1 \to \tilde{\lambda}_1$$
$$\lambda_n \to \lambda_n \qquad \tilde{\lambda}_n \to \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1$$

- On-shell condition:

$$(\hat{k}_1)^\mu(\hat{k}_1)_\mu = (\hat{k}_1)^{\alpha\dot{\alpha}}(\hat{k}_1)_{\dot{\alpha}\alpha} = \langle(\lambda_1 + z\lambda_n)(\lambda_1 + z\lambda_n)\rangle[1\,1] = 0$$

similarly, $\hat{k}_n^2 = 0$

- Momentum conservation:

$$\hat{k}_1 + \hat{k}_n = (\lambda_1 + z\lambda_n)\tilde{\lambda}_1 + \lambda_n(\tilde{\lambda}_n - z\tilde{\lambda}_1) = k_1 + k_n$$

MHV example

- Apply this shift to the Parke-Taylor (MHV) amplitudes:

$$A_n(z=0) = A_n^{jn,\,\mathrm{MHV}} = \frac{\langle j\,n\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

- Under the (n,1) shift: $\lambda_1 \to \lambda_1 + z\lambda_n \qquad \tilde{\lambda}_n \to \tilde{\lambda}_n - z\tilde{\lambda}_1$

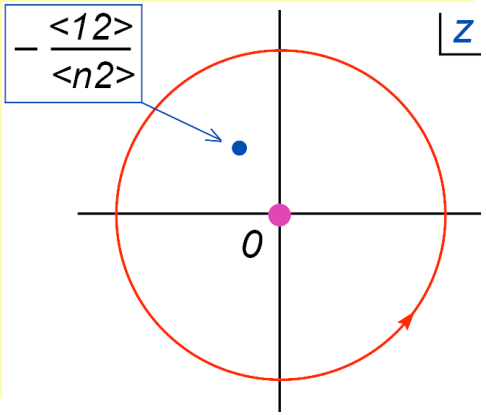
$$\langle n\,1\rangle = \lambda_n\lambda_1 \to \lambda_n(\lambda_1 + z\lambda_n) = \langle n\,1\rangle + z\langle n\,n\rangle = \langle n\,1\rangle$$

$$\langle 1\,2\rangle = \lambda_1\lambda_2 \to (\lambda_1 + z\lambda_n)\lambda_2 = \langle 1\,2\rangle + z\langle n\,2\rangle$$

- So $$A_n(z) = \frac{\langle j\,n\rangle^4}{(\langle 1\,2\rangle + z\langle n\,2\rangle)\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

- Consider: $$\frac{1}{2\pi i}\oint_C dz\,\frac{A_n(z)}{z}$$

- 2 poles, opposite residues



MHV example (cont.)

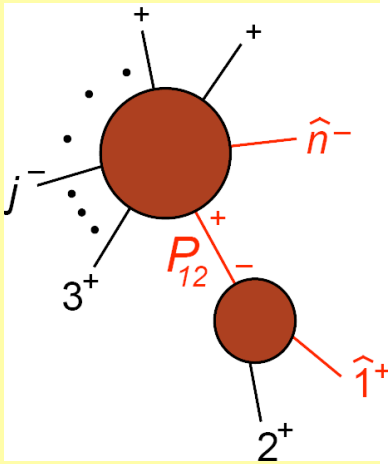
- MHV amplitude obeys: $$A_n(0) = -\operatorname*{Res}_{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}} \frac{A_n(z)}{z}$$

- Compute residue using factorization
- At $$z = -\frac{\langle 1\,2\rangle}{\langle n\,2\rangle} = -\frac{\langle 1\,2\rangle[2\,1]}{\langle n\,2\rangle[2\,1]} = -\frac{s_{12}}{\langle n^-|(1+2)|1^-\rangle}$$

  kinematics are complex collinear

$$\langle \hat{1}\,2\rangle = \langle 1\,2\rangle + z\langle n\,2\rangle = 0 \qquad [\hat{1}\,2] = [1\,2] \neq 0$$

$$s_{\hat{1}2} = \langle \hat{1}\,2\rangle[2\,\hat{1}] = 0$$



- so $$-\operatorname*{Res}_{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}} \frac{A_n(z)}{z} = A_{n-1}(\hat{P}_{12}^+, 3^+, \ldots, j^-, \ldots, n^-) \times \left[-\operatorname*{Res}_{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}} \frac{1}{z}\frac{1}{\hat{P}_{12}^2(z)}\right] A_3(\hat{1}^+, 2^+, -\hat{P}_{12}^-)$$

note
$A_3(+,+,+) = 0$

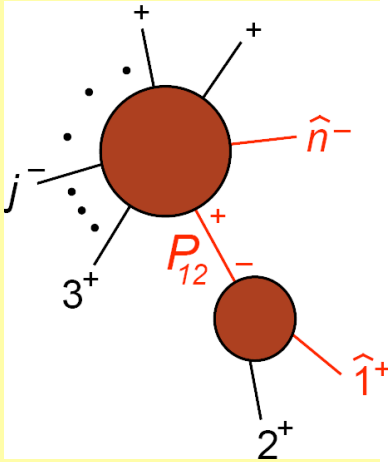
Evaluate the ingredients

- Since $\hat{P}_{12}^2(z) \;=\; (k_1 + k_2 + z\lambda_n\tilde{\lambda}_1)^2 = s_{12} + z\langle n^-|(1+2)|1^-\rangle$

$$-\operatorname*{Res}_{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}} \frac{1}{z}\frac{1}{\hat{P}_{12}^2(z)} \;=\; -\frac{\langle n^-|(1+2)|1^-\rangle}{s_{12}}\frac{1}{\langle n^-|(1+2)|1^-\rangle} = \frac{1}{s_{12}}$$

- So

$$\boxed{A_n(0) \;=\; A_{n-1}(\hat{P}_{12}^+, 3^+, \ldots, j^-, \ldots, n^-)\frac{1}{s_{12}}A_3(\hat{1}^+, 2^+, -\hat{P}_{12}^-)}$$



- Check this explicitly:

$$A_n(0) \;=\; \frac{\langle j\,\hat{n}\rangle^4}{\langle \hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,\hat{n}\rangle\langle \hat{n}\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[\hat{1}\,2]^3}{[2\,\hat{P}][\hat{P}\,\hat{1}]}$$

$$= \frac{\langle j\,n\rangle^4}{\langle \hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,n\rangle\langle n\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[1\,2]^3}{[2\,\hat{P}][\hat{P}\,1]}$$

MHV check (cont.)

- Using

$$\begin{aligned}\langle n\,\hat{P}\rangle[\hat{P}\,2] &= \langle n^-|(1+2)|2^-\rangle + z\langle n\,n\rangle[1\,2] = \langle n\,1\rangle[1\,2]\\ \langle 3\,\hat{P}\rangle[\hat{P}\,1] &= \langle 3^-|(1+2)|1^-\rangle + z\langle 3\,n\rangle[1\,1] = \langle 3\,2\rangle[2\,1]\end{aligned}$$

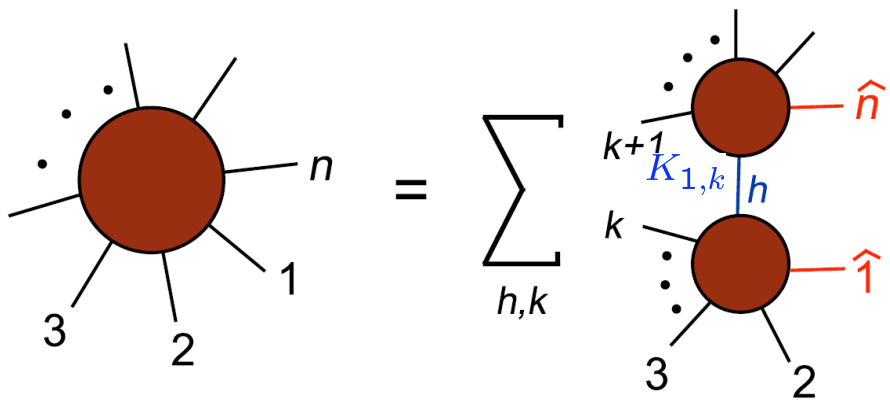
one confirms

$$\begin{aligned}A_n(0) &= \frac{\langle j\,n\rangle^4}{\langle\hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,n\rangle\langle n\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[1\,2]^3}{[2\,\hat{P}][\hat{P}\,1]}\\ &= \frac{\langle j\,n\rangle^4[1\,2]^3}{(\langle 1\,2\rangle[2\,1])([1\,2]\langle 2\,3\rangle)(\langle n\,1\rangle[1\,2])\langle 3\,4\rangle\cdots\langle n-1,n\rangle}\\ &= \frac{\langle j\,n\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n-1,n\rangle\langle n\,1\rangle}\\ &= A_n^{jn,\,\mathrm{MHV}}\end{aligned}$$

The general case

Britto, Cachazo, Feng, hep-th/0412308

$$A_n(1,2,\ldots,n) = \sum_{h=\pm}\sum_{k=2}^{n-2} A_{k+1}(\hat{1},2,\ldots,k,-\hat{K}^{-h}_{1,k}) \times \frac{i}{K^2_{1,k}} A_{n-k+1}(\hat{K}^{h}_{1,k},k+1,\ldots,n-1,\hat{n})$$

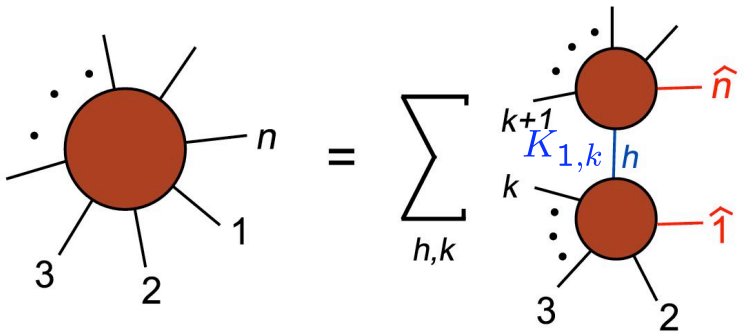


A_{k+1} and A_{n-k+1} are on-shell tree amplitudes with fewer legs, evaluated with 2 momenta shifted by a **complex** amount

Momentum shift

Shift for k^{th} term comes from setting $z = z_k$, where

$$z_k = -\frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle}$$



is the solution to

$$\hat{K}_{1,k}^2(z) = 0 = (K_{1,k} + z\lambda_n\tilde{\lambda}_1)^2 = K_{1,k}^2 + z\lambda_n^a (K_{1,k})_{a\dot{a}}\tilde{\lambda}_1^{\dot{a}}$$

plugging in, shift is:

$$\hat{\lambda}_1 = \lambda_1 - \frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle}\lambda_n \qquad \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1$$

$$\hat{\lambda}_n = \lambda_n \qquad \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n + \frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle}\tilde{\lambda}_1$$

Proof of on-shell recursion relations

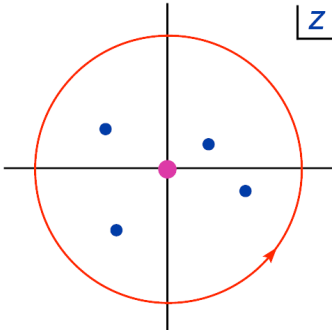
Britto, Cachazo, Feng, Witten, hep-th/0501052

Same analysis as above – Cauchy's theorem + **amplitude factorization**

Let complex momentum shift depend on z. Use analyticity in z.

$$\hat{\lambda}_1 = \lambda_1 + z\lambda_n \qquad \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1$$
$$\hat{\lambda}_n = \lambda_n \qquad \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1$$

$$\Rightarrow \quad A(0) \;\rightarrow\; A(z)$$

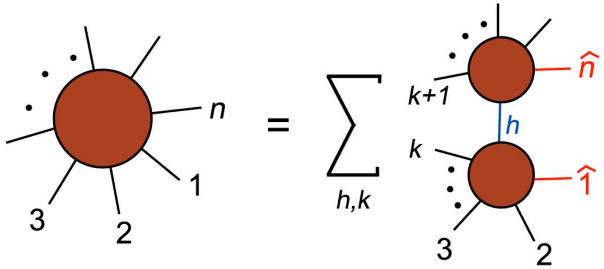


Cauchy: If $A(\infty) = 0$ then

$$0 = \frac{1}{2\pi i}\oint dz\,\frac{A(z)}{z} = A(0) + \sum_k \mathrm{Res}\left[\frac{A(z)}{z}\right]\Big|_{z=z_k}$$

poles in z: physical factorizations $\hat{K}^2_{1,k} = 0$

residue at $z_k = -\frac{K^2_{1,k}}{\langle n^-|K_{1,k}|1^-\rangle}$ = [k^{th} term]



To show: $A(\infty) = 0$

Britto, Cachazo, Feng, Witten, hep-th/0501052

Propagators:

$$\frac{1}{\widehat{K}^2_{1,k}(z)} = \frac{1}{K^2_{1,k} + z\lambda^a_n (K_{1,k})_{a\dot{a}} \tilde{\lambda}^{\dot{a}}_1} \sim \frac{1}{z}$$

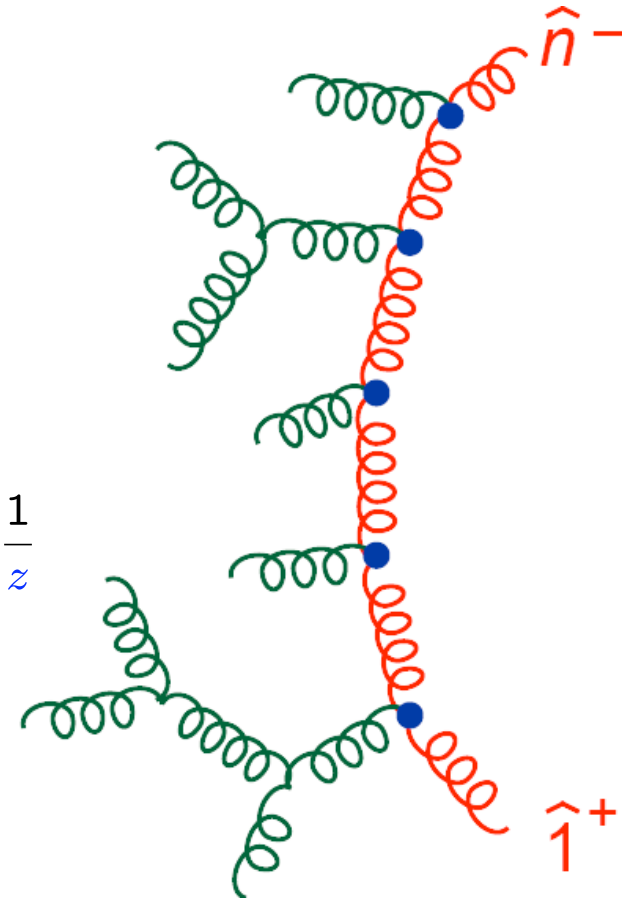
3-point vertices: $\propto \ \widehat{k}^\mu(z) \propto z$

Polarization vectors:

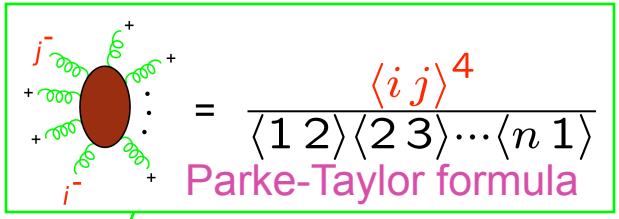
$$\not{\epsilon}^+_1 \ \propto \ \frac{\tilde{\lambda}_1 \lambda_q}{\langle \lambda_1 \, \lambda_q \rangle} \ \propto \ \frac{1}{z} \qquad \not{\epsilon}^-_n \ \propto \ \frac{\lambda_n \tilde{\lambda}_q}{\langle \tilde{\lambda}_n \, \tilde{\lambda}_q \rangle} \ \propto \ \frac{1}{z}$$

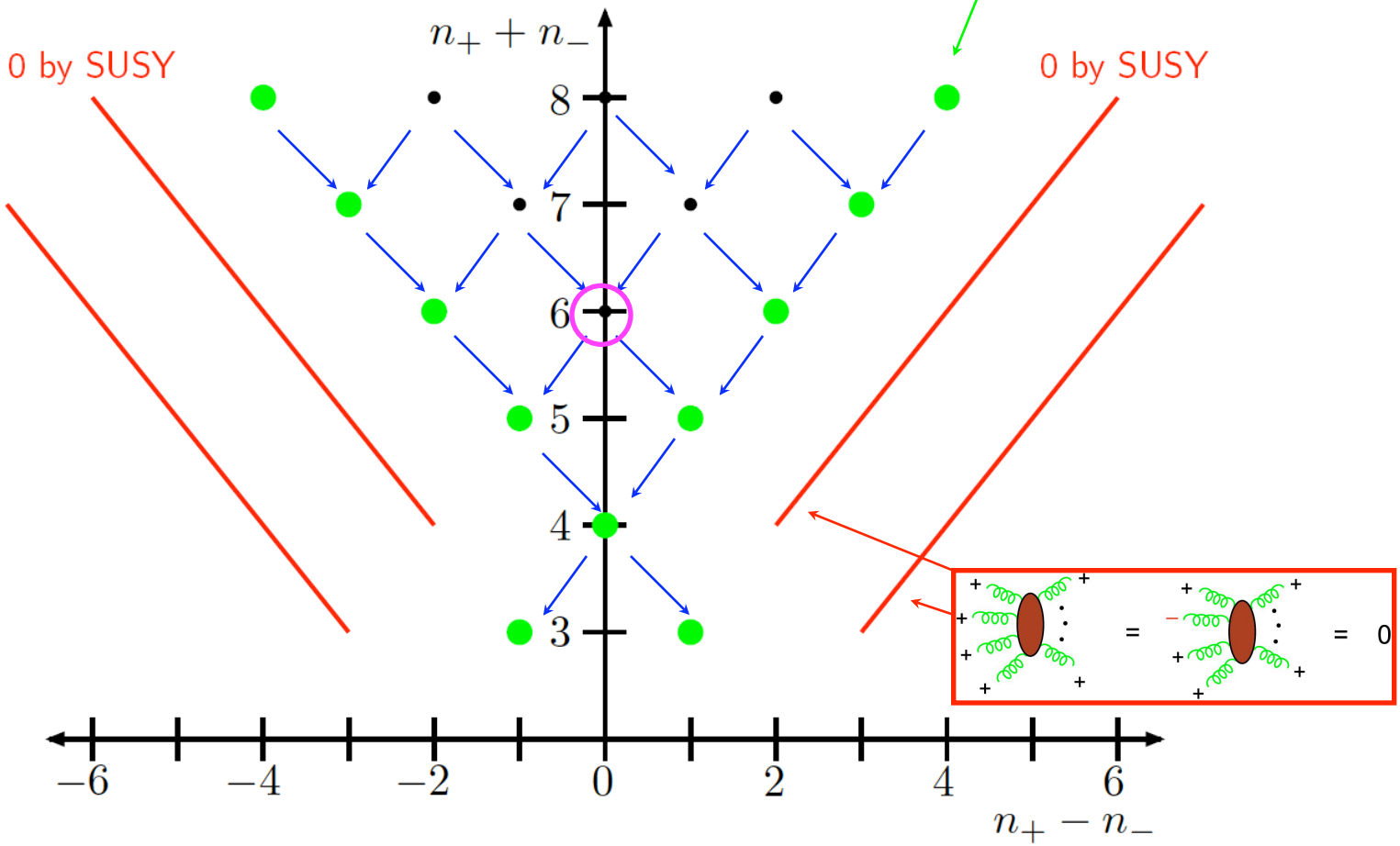
Total:

$$\frac{1}{z} \times \left(z \frac{1}{z} \right)^r z \times \frac{1}{z} = \frac{1}{z}$$



Initial data





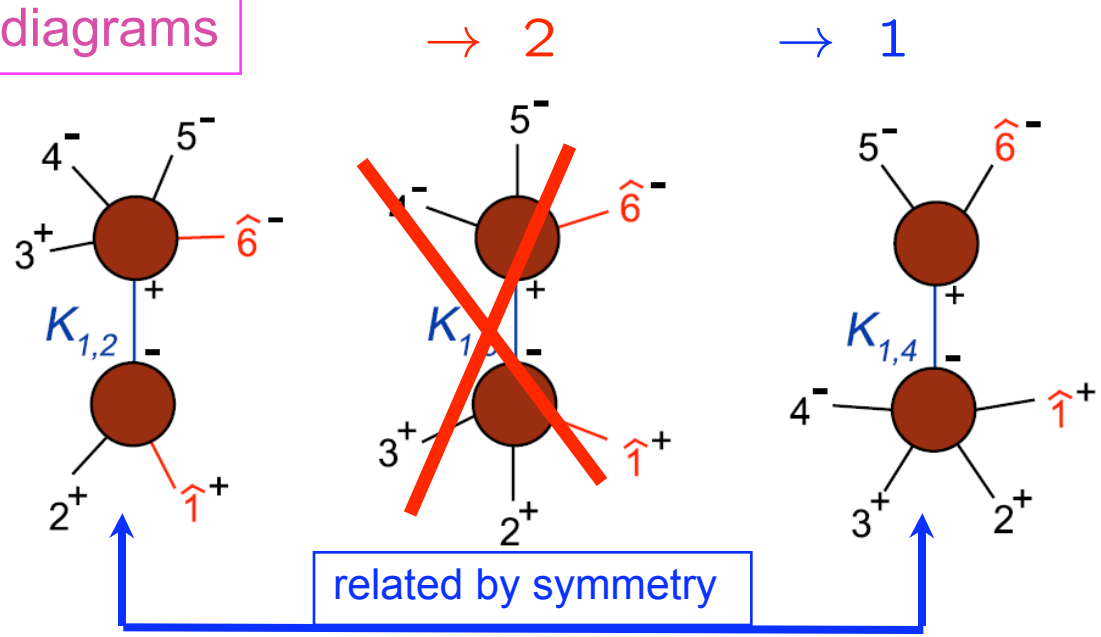
A 6-gluon example

220 Feynman diagrams for *gggggg*

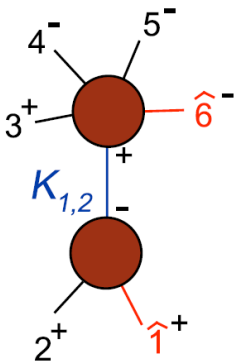
Helicity + color + MHV results + symmetries

$\Rightarrow$ only $\boxed{A_6(1^+,2^+,3^+,4^-,5^-,6^-)}$, $A_6(1^+,2^+,3^-,4^+,5^-,6^-)$

3 BCF diagrams



The one $A_6(1^+,2^+,3^+,4^-,5^-,6^-)$ diagram



$$= -\frac{i}{s_{12}} \frac{[\hat{1}\,2]^3}{[2\,\hat{K}][\hat{K}\,\hat{1}]} \frac{[\hat{K}\,3]^3}{[3\,4][4\,5][5\,\hat{6}][\hat{6}\,\hat{K}]}$$

$$= -\frac{i}{s_{12}} \frac{[1\,2]^3}{([2\,\hat{K}]\langle\hat{K}\,6\rangle)(\langle 6\,\hat{K}\rangle[\hat{K}\,1])} \frac{(\langle 6\,\hat{K}\rangle[\hat{K}\,3])^3}{[3\,4][4\,5][5\,\hat{6}]([\hat{6}\,\hat{K}]\langle\hat{K}\,6\rangle)}$$

$$= i\,\frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\langle 1\,2\rangle[3\,4][4\,5]s_{612}\langle 2^-|(6+1)|5^-\rangle}$$

$$\begin{aligned}
\langle 6\,\hat{K}\rangle[\hat{K}\,a] &= \langle 6\,1\rangle\,[1\,a] + \langle 6\,2\rangle\,[2\,a] \\
&= \langle 6^-|(1+2)|a^-\rangle
\end{aligned}$$

$$[5\,\hat{6}] = [5\,6] + \frac{s_{12}[5\,1]}{\langle 6\,2\rangle[2\,1]} = \frac{\langle 5^+|(6+1)|2^+\rangle}{\langle 6\,2\rangle}$$

$$[\hat{6}\,\hat{K}]\langle\hat{K}\,6\rangle = \langle 6^+|(1+2)|6^+\rangle + s_{12} = s_{612}$$

Simple final form

$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,s_{612}\langle 2^-|(6+1)|5^-\rangle} + \frac{\langle 4^-|(5+6)|1^-\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5\,6]\,[6\,1]\,s_{561}\langle 2^-|(6+1)|5^-\rangle}$$

Simpler than form found in 1980s Mangano, Parke, Xu (1988)
despite (because of?) spurious singularities $\langle 2^-|(6+1)|5^-\rangle$

$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{([1\,2]\,\langle 4\,5\rangle\,\langle 6^-|(1+2)|3^-\rangle)^2}{s_{61}s_{12}s_{34}s_{45}s_{612}} + \frac{([2\,3]\,\langle 5\,6\rangle\,\langle 4^-|(2+3)|1^-\rangle)^2}{s_{23}s_{34}s_{56}s_{61}s_{561}} + \frac{s_{123}\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6^-|(1+2)|3^-\rangle\langle 4^-|(2+3)|1^-\rangle}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}}$$

Relative simplicity even more striking for *n>6*

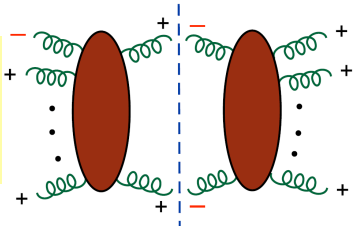
Bern, Del Duca, LD, Kosower (2004)

On-shell recursion at one loop

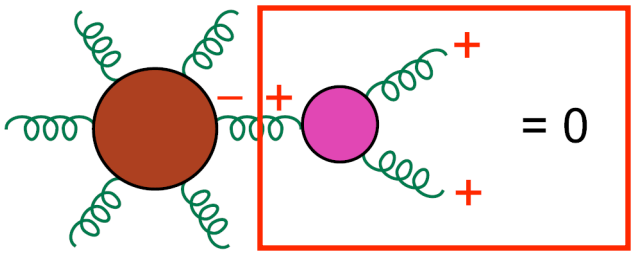
Bern, LD, Kosower, hep-th/0501240, hep-th/0505055

- **Same techniques** can be used to compute one-loop amplitudes -- which are much harder to obtain by other methods than are trees.

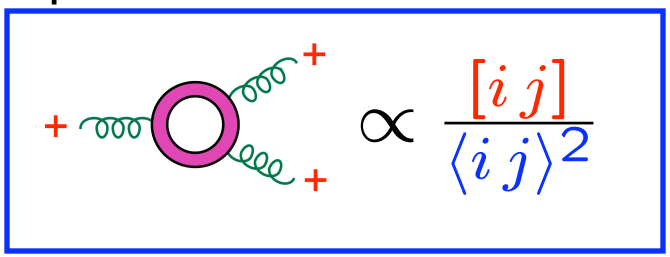
- First consider special tree-like one-loop amplitudes with no cuts, only poles: $A_n^{1-\text{loop}}(1^\pm, 2^+, 3^+, \ldots, n^+)$



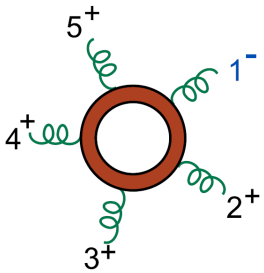
- **New features** arise compared with tree case due to different collinear behavior of loop amplitudes:



but



A one-loop pole analysis

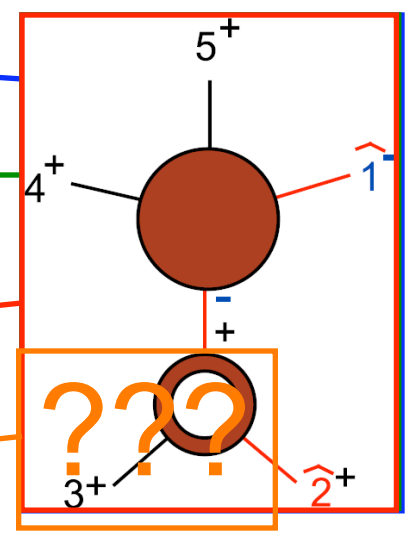


$$= -\frac{[2\,5]^3}{[5\,1][1\,2]\langle 3\,4\rangle^2} + \frac{\langle 1\,4\rangle^3[4\,5]\langle 3\,5\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle^2\langle 4\,5\rangle^2} + \frac{\langle 1\,3\rangle^3[2\,3]\langle 2\,4\rangle}{\langle 2\,3\rangle^2\langle 3\,4\rangle^2\langle 4\,5\rangle\langle 5\,1\rangle}$$

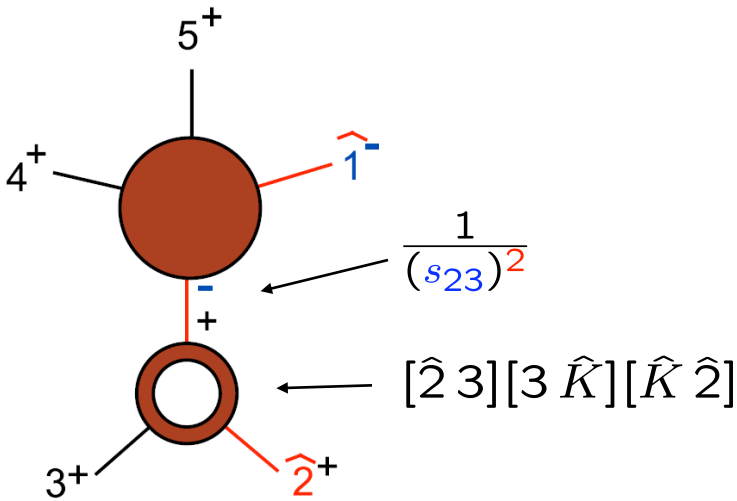
Bern, LD, Kosower (1993)

under shift $\hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 - z\tilde{\lambda}_2 \qquad \hat{\lambda}_2 = \lambda_2 + z\lambda_1$ plus partial fraction

$$\Rightarrow \quad -\frac{[2\,5]^3}{([5\,1] - z[5\,2])[1\,2]\langle 3\,4\rangle^2} + \frac{\langle 1\,4\rangle^3[4\,5]\langle 3\,5\rangle}{\langle 1\,2\rangle(\langle 2\,3\rangle + z\langle 1\,3\rangle)\langle 3\,4\rangle^2\langle 4\,5\rangle^2} - \frac{\langle 1\,3\rangle^2[2\,3]\langle 1\,2\rangle\langle 3\,4\rangle}{(\langle 2\,3\rangle + z\langle 1\,3\rangle)^2\langle 3\,4\rangle^2\langle 4\,5\rangle\langle 5\,1\rangle} - \frac{\langle 1\,3\rangle^2[2\,3]\langle 1\,4\rangle}{(\langle 2\,3\rangle + z\langle 1\,3\rangle)\langle 3\,4\rangle^2\langle 4\,5\rangle\langle 5\,1\rangle}$$



Underneath the double pole



Missing diagram should be related, but suppressed by factor of s_{23}

Don't know collinear behavior at this level, must guess the correct suppression factor:

$$s_{23}\,\mathcal{S}(a,\hat{K}^+,b)\,\mathcal{S}(c,(-\hat{K})^-,d)$$

in terms of universal eikonal factors for soft gluon emission

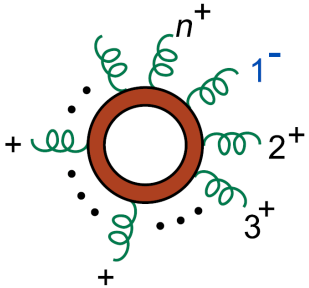
$$\mathcal{S}(a,s^+,b) = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle}$$

$$\mathcal{S}(a,s^-,b) = -\frac{[a\,b]}{[a\,s][s\,b]}$$

Here, multiplying 3rd diagram by $s_{23}\,\mathcal{S}(\hat{1},\hat{K}^+,4)\,\mathcal{S}(3,(-\hat{K})^-,\hat{2})$ gives the correct missing term!

A one-loop all-*n* recursion relation

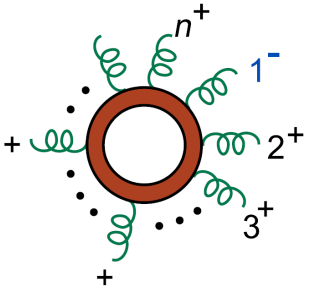
Same suppression factor works in the case of *n* external legs!



$$
\begin{aligned}
&A_n^{(1)}(1^-,2^+,\ldots,n^+) \\
&\quad = A_{n-1}^{(1)}(4^+,5^+,\ldots,n^+,\hat{1}^-,\hat{K}_{23}^+)\,\frac{i}{K_{23}^2}\,A_3^{(0)}(\hat{2}^+,3^+,-\hat{K}_{23}^-) \\
&\quad + \sum_{j=4}^{n-1} A_{n-j+2}^{(0)}((j+1)^+,5^+,\ldots,n^+,\hat{1}^-,\hat{K}_{2\ldots j}^-)\,\frac{i}{K_{2\ldots j}^2}\,A_j^{(1)}(\hat{2}^+,3^+,\ldots,j^+,-\hat{K}_{2\ldots j}^+) \\
&\quad + A_{n-1}^{(0)}(4^+,5^+,\ldots,n^+,\hat{1}^-,\hat{K}_{23}^-)\,\frac{i}{(K_{23}^2)^2}\,V_3^{(1)}(\hat{2}^+,3^+,-\hat{K}_{23}^+) \\
&\qquad\qquad \times\Big(1 + K_{23}^2\,\mathcal{S}^{(0)}(\hat{1},\hat{K}_{23}^+,4)\,\mathcal{S}^{(0)}(3,-\hat{K}_{23}^-,\hat{2})\Big)
\end{aligned}
$$

Know it works because results agree with Mahlon, hep-ph/9312276, though much shorter formulae are obtained from this relation

Solution to recursion relation



$$A_n^{(1)}(1^-,2^+,3^+,\ldots,n^+) = \frac{i}{3}\frac{T_1+T_2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,,$$

where

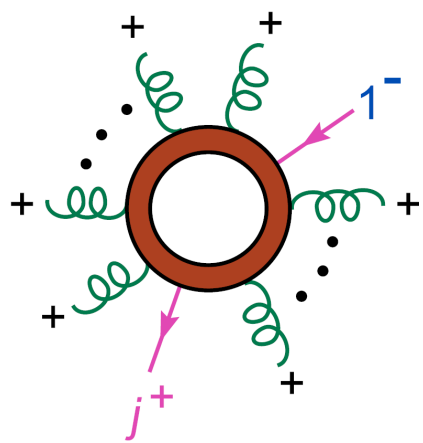
$$T_1 = \sum_{l=2}^{n-1}\frac{\langle 1\,l\rangle\,\langle 1\,(l+1)\rangle\,\langle 1^-|\,\not{K}_{l,l+1}\not{K}_{(l+1)\cdots n}\,|1^+\rangle}{\langle l\,(l+1)\rangle}\,,$$

$$\begin{aligned}
T_2 = \sum_{l=3}^{n-2}\sum_{p=l+1}^{n-1} & \frac{\langle (l-1)\,l\rangle}{\langle 1^-|\,\not{K}_{(p+1)\cdots n}\not{K}_{l\cdots p}\,|(l-1)^+\rangle\,\langle 1^-|\,\not{K}_{(p+1)\cdots n}\not{K}_{l\cdots p}\,|l^+\rangle} \\
& \times\frac{\langle p\,(p+1)\rangle}{\langle 1^-|\,\not{K}_{2\cdots(l-1)}\not{K}_{l\cdots p}\,|p^+\rangle\,\langle 1^-|\,\not{K}_{2\cdots(l-1)}\not{K}_{l\cdots p}\,|(p+1)^+\rangle} \\
& \times\left\langle 1^-\right|\,\not{K}_{l\cdots p}\not{K}_{(p+1)\cdots n}\left|1^+\right\rangle^3 \\
& \times\frac{\langle 1^-|\,\not{K}_{2\cdots(l-1)}[\mathcal{F}(l,p)]^2\not{K}_{(p+1)\cdots n}\,|1^+\rangle}{s_{l\cdots p}}\,.
\end{aligned}$$

$$\mathcal{F}(l,p) = \sum_{i=l}^{p-1}\sum_{m=i+1}^{p}\not{k}_i\not{k}_m$$

External fermions too

Can similarly write down recursion relations
for the finite, cut-free amplitudes with 2 external fermions:



and the solutions are just as compact

Gives the complete set of finite, cut-free, QCD loop amplitudes
(at 2 loops or more, all helicity amplitudes have cuts, diverge)

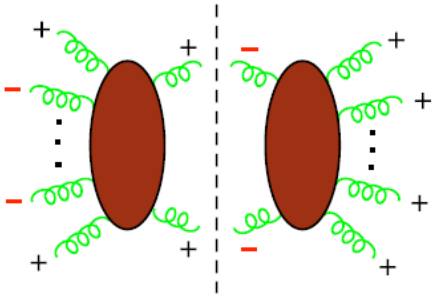
Loop amplitudes with cuts

- Recently extended same recursive technique (combined with unitarity) to loop amplitudes with cuts (hep-ph/0507005)
- Here rational-function terms contain "spurious singularities", e.g. $\sim \frac{\ln(r) + 1 - r}{(1-r)^2}, \quad r = s_2/s_1$
- accounting for them properly yields simple "overlap diagrams" in addition to recursive diagrams
- No loop integrals required to bootstrap the rational functions from the cuts and lower-point amplitudes
- Tested method on 5-point amplitudes, used it to compute $A_6(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$, $A_7(1^-, 2^-, 3^+, 4^+, 5^+, 6^+, 7^+)$

Revenge of the Analytic *S*-matrix?

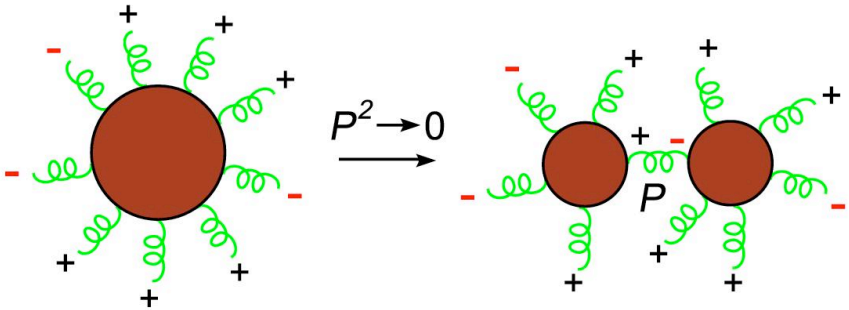
Reconstruct scattering amplitudes **directly** from **analytic properties**

- Branch cuts



Chew, Mandelstam; Eden, Landshoff, Olive, Polkinghorne; … (1960s)

- Poles



Analyticity fell out of favor in 1970s with rise of QCD; to resurrect it for computing **perturbative** QCD amplitudes seems **deliciously ironic**!

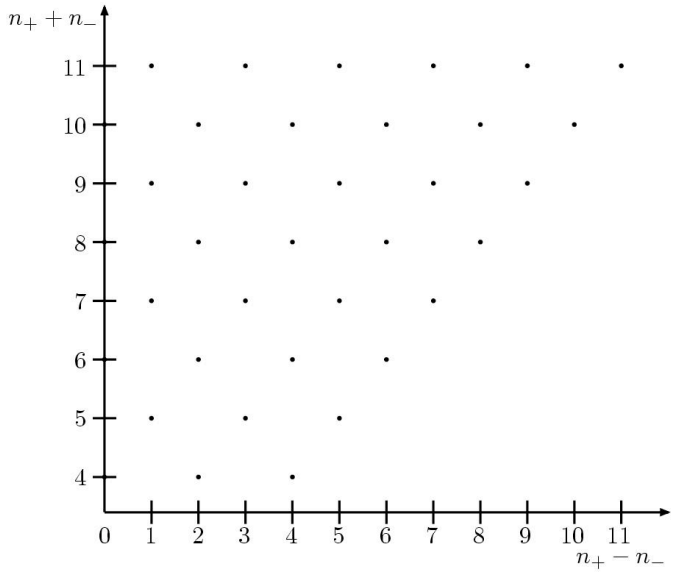
Conclusions

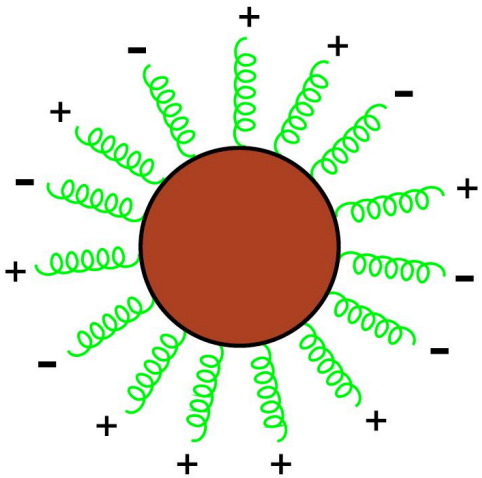
- On-shell recursion relations a very efficient way to compute multi-leg tree amplitudes in gauge theory
- Development an indirect spinoff from twistor string theory
- Can extend relations to special loop amplitudes
  -- with some guesswork
- Method still very efficient; compact solutions found for all finite, cut-free loop amplitudes in QCD
- Recently extended same technique (combined with unitarity) to some of the more general loop amplitudes with cuts, needed for NLO corrections to LHC processes
- Prospects look very good for attacking a wide range of multi-parton processes in this way

Why does it all work?

In mathematics you don't understand things.
You just get used to them.

March of the *n*-gluon helicity amplitudes





n_+ positive-helicity gluons

n_- negative-helicity gluons

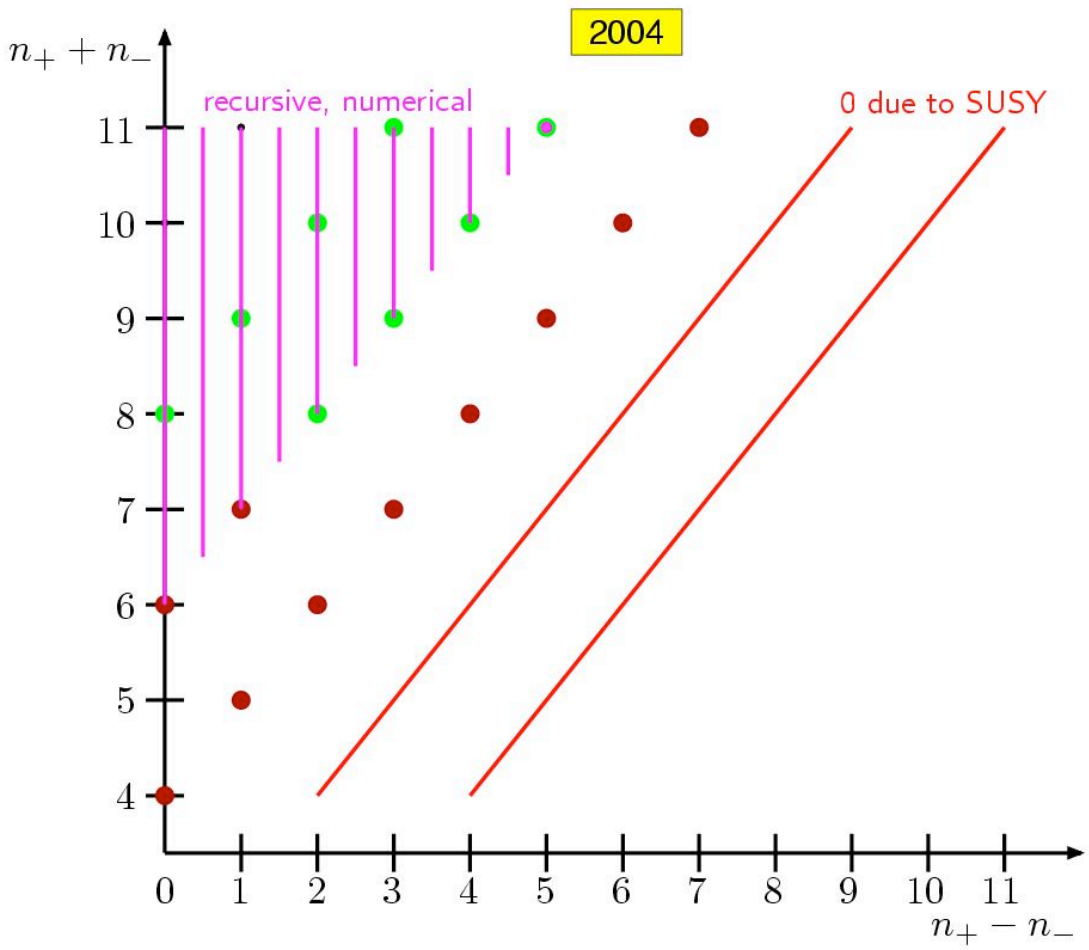
$n = n_+ + n_- \geq 4$

$n_+ \geq n_-$ by parity

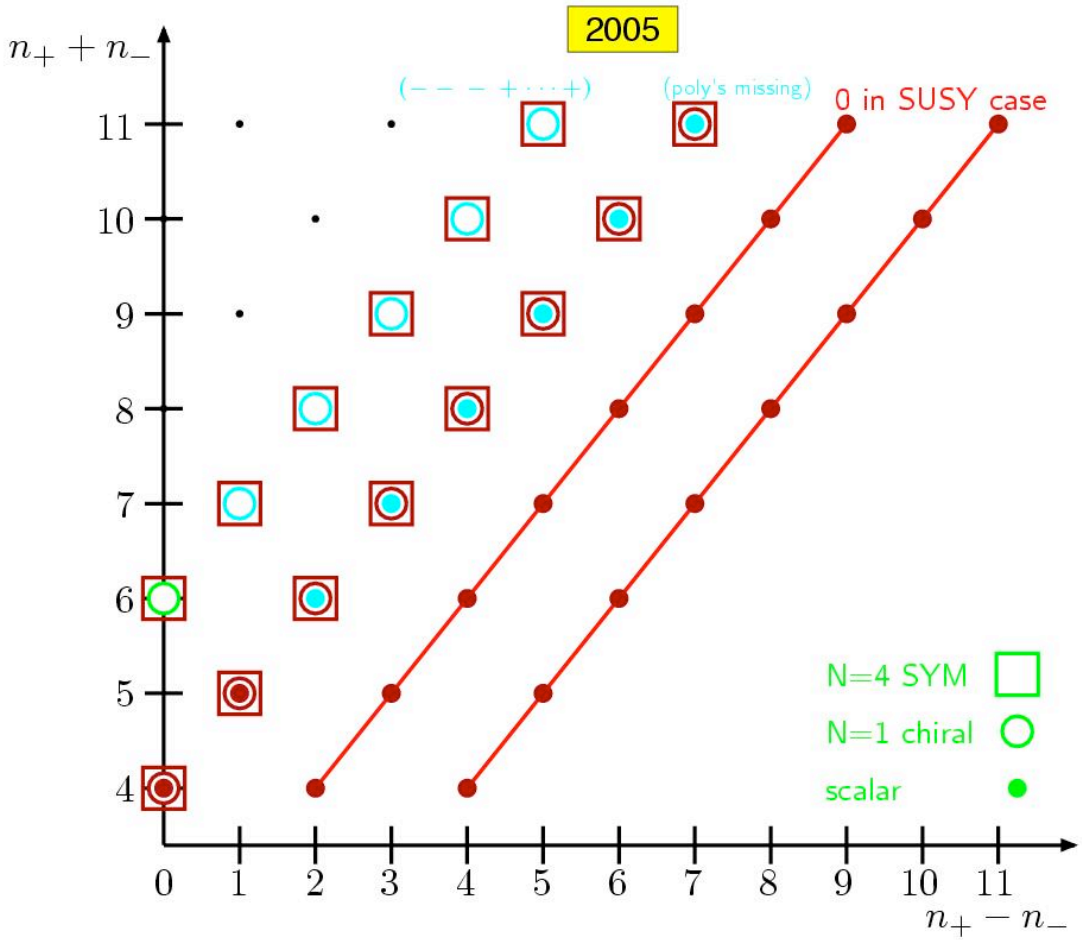
At 1-loop, QCD decomposable into

N=4 SYM, N=1 chiral, scalar contributions

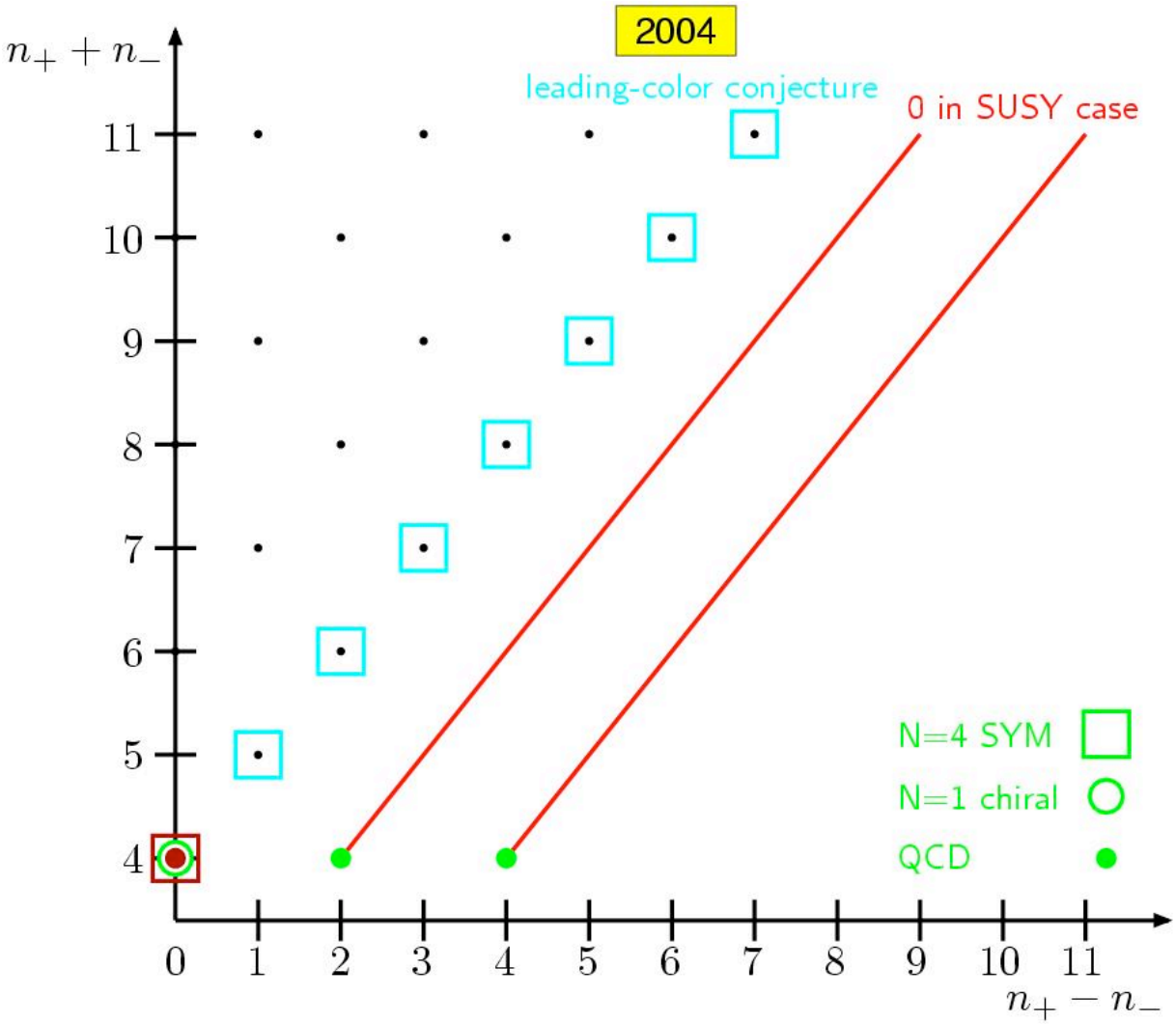
March of the tree amplitudes



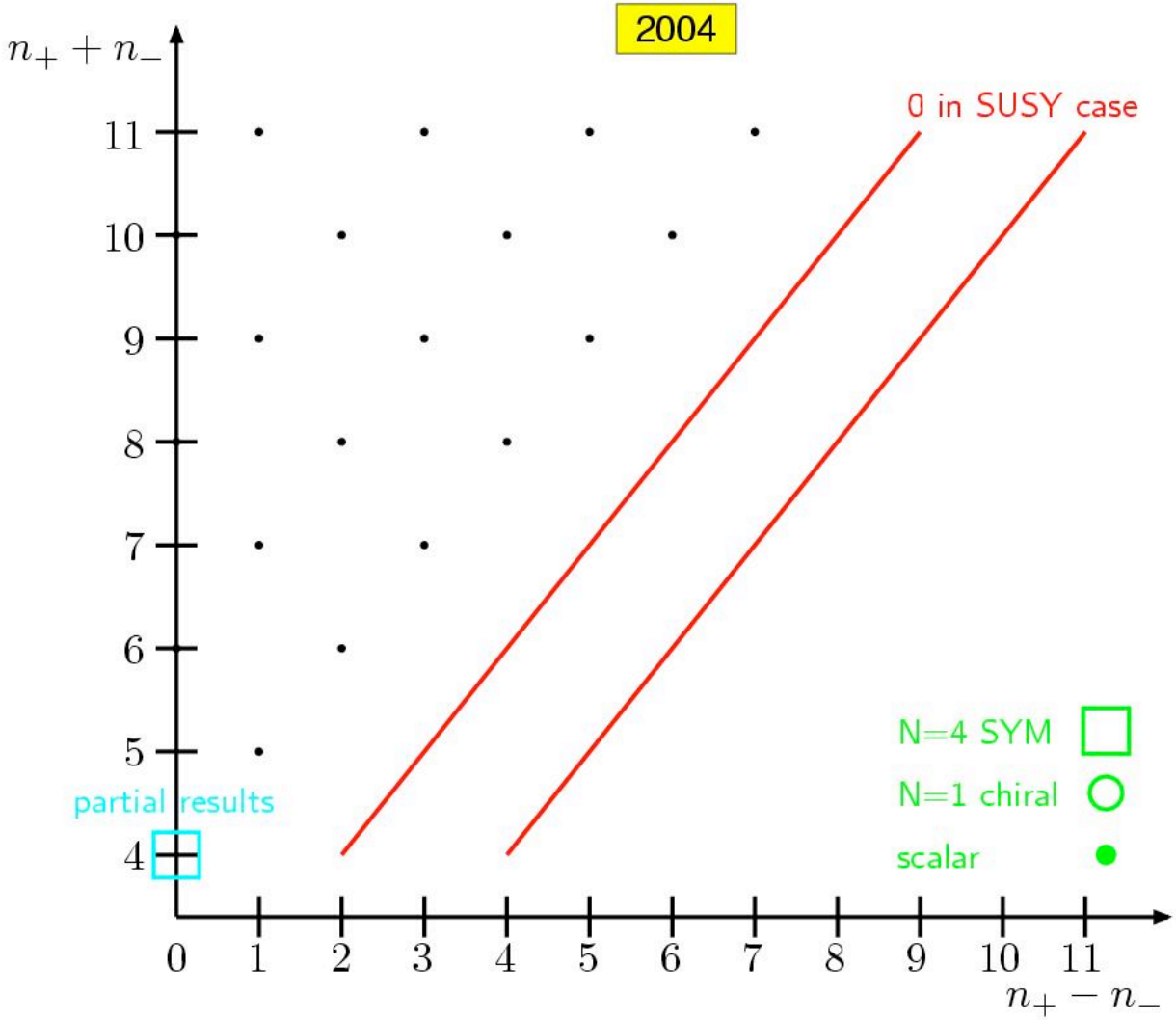
March of the 1-loop amplitudes



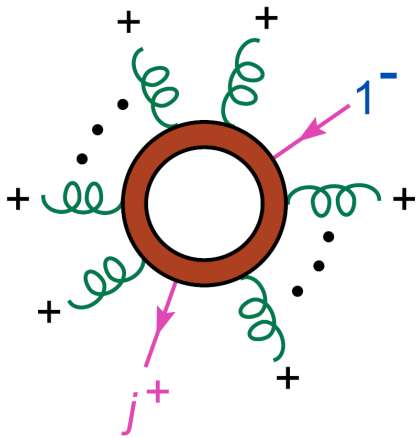
March of the 2-loop amplitudes



March of the 3-loop amplitudes



Fermionic solutions



$$A_n^{L-s}(1_f^-, 2^+, \ldots, j_f^+, \ldots, n^+) = \frac{i}{2} \frac{\langle 1\, j\rangle \sum_{l=3}^{n-1} \langle 1^- | \not{K}_{2\ldots l} \not{k}_l | 1^+\rangle}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$$

and

$$A_n^s(j_f^+) = \frac{i}{3} \frac{S_1 + S_2}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle n\, 1\rangle},$$

where

$$S_1 = \sum_{l=j+1}^{n-1} \frac{\langle j\, l\rangle \langle 1\,(l+1)\rangle \langle 1^- | \not{K}_{l,l+1} \not{K}_{(l+1)\cdots n} | 1^+\rangle}{\langle l\,(l+1)\rangle},$$

$$\begin{aligned} S_2 = \sum_{l=j+1}^{n-2} \sum_{p=l+1}^{n-1} & \frac{\langle (l-1)\, l\rangle}{\langle 1^- | \not{K}_{(p+1)\cdots n} \not{K}_{l\cdots p} | (l-1)^+\rangle \langle 1^- | \not{K}_{(p+1)\cdots n} \not{K}_{l\cdots p} | l^+\rangle} \\ & \times \frac{\langle p\,(p+1)\rangle}{\langle 1^- | \not{K}_{2\cdots(l-1)} \not{K}_{l\cdots p} | p^+\rangle \langle 1^- | \not{K}_{2\cdots(l-1)} \not{K}_{l\cdots p} | (p+1)^+\rangle} \\ & \times \langle 1^- | \not{K}_{l\cdots p} \not{K}_{(p+1)\cdots n} | 1^+\rangle^2 \langle j^- | \not{K}_{l\cdots p} \not{K}_{(p+1)\cdots n} | 1^+\rangle \\ & \times \frac{\langle 1^- | \not{K}_{2\cdots(l-1)} [\mathcal{F}(l,p)]^2 \not{K}_{(p+1)\cdots n} | 1^+\rangle}{s_{l\cdots p}}, \end{aligned}$$

$$\mathcal{F}(l,p) = \sum_{i=l}^{p-1} \sum_{m=i+1}^{p} \not{k}_i \not{k}_m$$