THE WARPED DEFORMED CONIFOLD

AND

ITS DUAL CASCADING

GAUGE THEORY

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ONASSIS LECTURES

Recommended reding:

C. Herzog, P. Ouyang, IK, hep-th/0205100

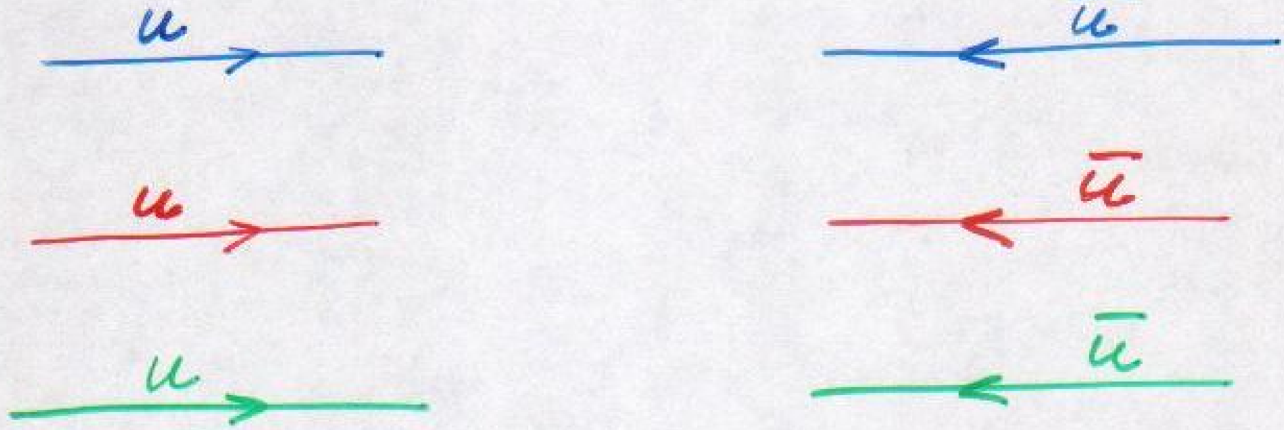
S. Gubser, C. Herzog, IK, hep-th/0405282

QCD: THE SU(3) GAUGE THEORY OF STRONG INTERACTIONS

Hadrons are made of quarks and gluons.

Each quark flavor (up, down, strange, ...)
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad u \quad\quad\quad d \quad\quad\quad\quad s$

and the corresponding anti-quark comes in THREE different colors, e.g.



up quarks $\quad\quad\quad\quad\quad\quad$ up antiquarks

The action contains the quark fields q_α^i; α is the flavor index, and $i = 1, 2, 3$ is the color index.

The local SU(3) symmetry is $q^i \to U^i{}_j(x)\, q^j$.

There are also the gluon fields (analogues of the photon in QED), $A_{\mu}{}^{i}{}_{j}$; $\mu = 0, 1, 2, 3$.

Roughly, a gluon has two colors:

Also, $\sum_{i=1}^{3} A_{\mu}{}^{i}{}_{i} = 0 \Rightarrow$ each A_μ is a traceless hermitian 3×3 matrix, hence has EIGHT independent color states.

The QCD Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \bar{q}_{\alpha j}\left(-i\gamma^{\mu}\partial_{\mu} + m_{\alpha}\right)q_{\alpha}^{j}$$

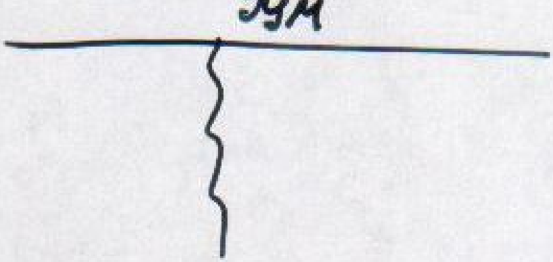
$$+\, g_{YM}\, \bar{q}_{\alpha i}\, \gamma^{\mu} A_{\mu}{}^{i}{}_{j}\, q_{\alpha}^{j} ,$$

where g_{YM} is the strong interaction coupling constant and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\, g_{YM}\,[A_\mu, A_\nu]$$

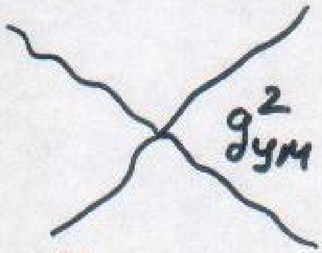
is the color field strength.

The quark-gluon interactions are summarized in the graphical Feynman rules



quark-gluon

3-gluon



4-gluon

Quantum effects render g_{YM} scale-dependent:

$$L \frac{\partial g_{YM}}{\partial L} = b_1 \frac{g_{YM}^3}{(4\pi)^2} + b_2 \frac{g_{YM}^5}{(4\pi)^4} + \dots$$

For the pure glue theory (no quarks) $b_1 = 11 \Rightarrow g_{YM}$ becomes small at short distances (high energies) Gross, Wilczek, Politzer

$$\frac{1}{g_{YM}^2} = \frac{11}{8\pi^2} \ln\left(L_0/L\right), \text{ for } L \ll L_0.$$

L_0 is the dynamically generated "QCD scale".

At short distances QCD becomes free (the asymptotic freedom).

At long distances, where hadron properties need to be calculated, the coupling is strong => ordinary perturbation theory is not useful!

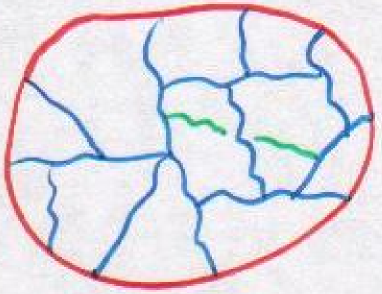
New techniques are needed for studying STRONGLY COUPLED GAUGE THEORIES.

A very fruitful proposal was made in 1974 by 't Hooft:

to replace 3 colors of QCD by a large number N. Then $N=3$ may be studied as the $N=\infty$ theory with corrections in powers of $1/N$.

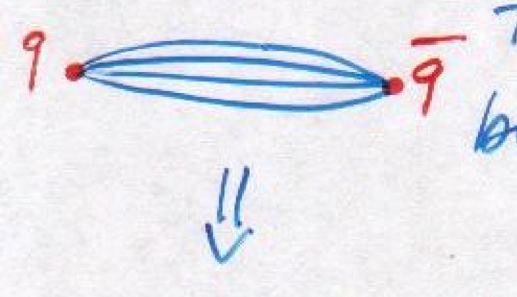
As $N \to \infty$, $g_{YM}^2 N$ has to be held fixed. In this 't Hooft limit, planar graphs dominate.

Consider a Wilson loop (the amplitude for a quark to propagate around a closed circuit):



The dominant graphs have the "planar" structure shown. Adding a green line (a non-planar correction) costs a multiplicative factor $\frac{1}{N^2}$ => the non-planar graphs are negligible as $N \to \infty$.

The sum over the planar graphs may be thought of as the theory of free open strings!



These open strings are nothing but the mesons of (large N) QCD.

The amplitude for a flux tube to "snap" via production of $q\bar{q}$ pair is of order $1/\sqrt{N}$.

To demonstrate CONFINEMENT one needs to prove that the flux tube (the QCD string) indeed forms and gives rise to the LINEAR $q\bar{q}$ potential (Wilson).

Thus, large N QCD is a good setting for studying confinement.

In addition to the open strings (mesons) QCD contains closed strings (glueballs).

These are the only physical particles in the gauge theory without dynamical quarks (pure glue theory).

Numerical studies of such theories on a lattice indicate that the 't Hooft large N limit is a good approximation to $N=3$.

In 2+1 dimensions Teper finds for each glueball mass

$$m_i = g_{YM}^2 N \left(a_i^{(0)} + a_i^{(1)} N^{-2} \right)$$ gives a good fit!

LOW-LYING GLUEBALL SPECTRUM IN 2+1 DIMENSIONAL PURE GLUE THEORY, FROM M. TEPER, "SU(N) GAUGE THEORIES IN 2+1 DIMENSIONS," hep-lat/9804008

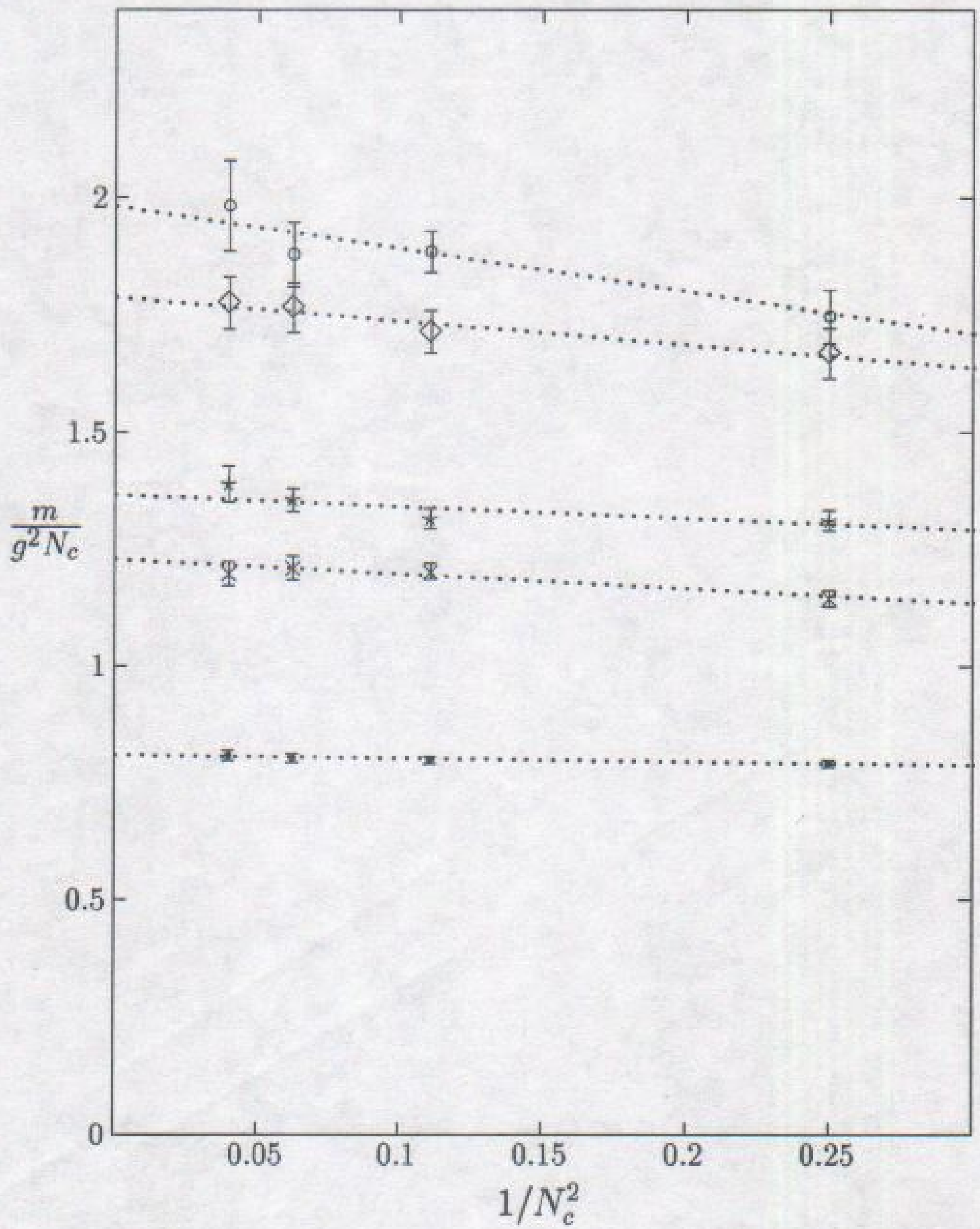


Figure 12: Some of the $C = +$ glueball masses for 2,3,4 and 5 colours, in units of $g^2 N_c$ and plotted against $1/N_c^2$: $0^{++}(\bullet)$, $0^{++*}(\times)$, $2^{++}(\star)$, $0^{-+}(\diamond)$, $1^{++}(\circ)$. The best linear extrapolations to the $N_c = \infty$ limit are also shown.

CHIRAL SYMMETRY BREAKING

QCD with N_f massless quarks;
$\mathcal{L}$ is invariant under independent flavor rotations of q_L^i and q_R^i.
$U(N_f)_L \times U(N_f)_R$.

$U(1)_A$ is broken by the axial anomaly (instantons) but this effect is weak for large N.

The vacuum expectation value

$$\langle \bar{q}_{Li} \, q_R^j \rangle = \Lambda^3 \delta_i^j$$

breaks the symmetry to $SU(N_f) \times U(1)_B$

$\Rightarrow$ there are $N_f^2 - 1$ Goldstone bosons

In real world $N_f \approx 3$ (3 light quarks)

$\Rightarrow$ an octet of light pseudoscalar mesons: π's, K's, η.

For large N, η' is light $\Rightarrow$ nonet.

DURING THE LAST SEVERAL YEARS NEW INSIGHTS INTO GAUGE THEORY HAVE BEEN OBTAINED BY EMBEDDING IT INTO STRING THEORY IN A NOVEL WAY.

Gauge theories can be made to live on multi-dimensional extended objects called D-branes.

These gauge theories are typically supersymmetric relatives of $SU(N)$ gauge theory discussed above.

The simplest example is to study 3-dimensional D-branes (the 3-branes) embedded into 9+1 dimensional superstring theory.

The gauge theory on 3-branes is the maximally supersymmetric 3+1 dimensional gauge theory.

In addition to gluons it contains 6 scalar fields and 4 fermions. Now, the coupling g_{YM} does not "flow" $\Rightarrow$ the theory is CONFORMAL.

In general, for Dp-branes

$$g_s \int_{S^{8-p}} * F_{p+2} = 2\kappa^2 \tau_p N ,$$

$$\tau_p = \frac{\sqrt{\pi}}{\kappa} \left(4\pi^2 \alpha'\right)^{\frac{3-p}{2}} ,$$

$\kappa = 8\pi^{7/2} g_s \alpha'^2$ is the gravitational constant

$$\kappa^2 = 8\pi G_N ;$$

For $p=3$ we found $g_s \int_{S^5} F_5 = 16\pi^4 (\alpha')^2 g_s N$.

For the solution $g_s \int_{S^5} F_5 = 4L^4 \times \pi^3$;

$$\frac{L^4}{(\alpha')^2} = 4\pi g_s N = g_{YM}^2 N \equiv \lambda ,$$

the 't Hooft coupling.

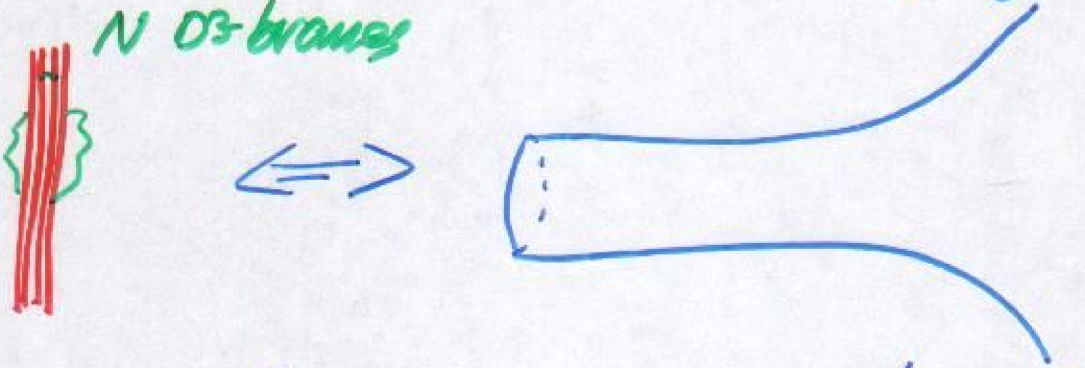
In the low-energy (small r) limit we expect duality between $\mathcal{N}=4$ $SU(N)$ SYM and the $AdS_5 \times S^5$ throat of the 3-brane, of radius L.

String σ-model corrections give powers of $\frac{\alpha'}{L^2} \sim \frac{1}{\sqrt{g_{YM}^2 N}}$

String loops give powers of $\frac{\kappa^2}{L^8} \sim \frac{1}{N^2}$.

MOTIVATION FOR AdS/CFT DUALITY
(MAXIMALLY SUPERSYMMETRIC CASE)

Compare a stack of many coincident D3-branes with the SUGRA solution carrying 5-form flux



$$ds^2 = H^{-\frac{1}{2}}(r)\, \eta_{\alpha\beta}\, dx^\alpha dx^\beta + H^{\frac{1}{2}}(r)\left(dr^2 + r^2 d\Omega_5^2\right)$$

$$\Phi = \text{const}; \quad H(r) = 1 + \frac{L^4}{r^4}.$$

$$g_s F_5 = d^4x \wedge dH^{-1} - r^5 H' \,\mathrm{vol}(S^5)$$

so that $F_5 = * F_5$

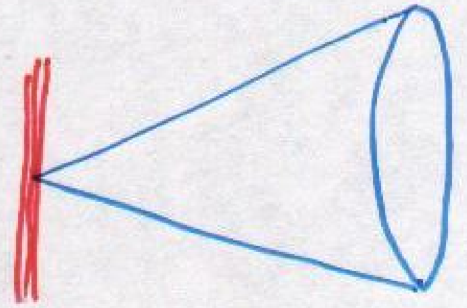
The Einstein eqn:

$$R_{MN} = \frac{g_s^2}{96} F_{MPQRS} F_N{}^{PQRS}$$ is obeyed.

L^4 is determined by requiring that there are N units of 5-form flux.

It is possible to break some of the SUSY without destroying conformal invariance.

Place D3-branes at the apex of a Ricci-flat 6-d cone Y_6, whose base is X_5 (Kachru, Silverstein; ...)



X_5 is an Einstein manifold:

$$R_{ij} = 4 g_{ij},$$

The metric of Y_6 is

$$ds_6^2 = dr^2 + r^2 g_{ij} d\theta^i d\theta^j.$$

The metric produced by N D3-branes is

$$ds^2 = H(r)^{-\frac{1}{2}} dx_{\parallel}^2 + H(r)^{\frac{1}{2}} (dr^2 + r^2 g_{ij} d\theta^i d\theta^j),$$

$$H(r) = 1 + \frac{L^4}{r^4}; \quad L^4 \sim g_s N \alpha'^2;$$

In the throat ($r \Rightarrow 0$) limit we find

$$ds^2 \to \frac{r^2}{L^2} dx_{\parallel}^2 + \frac{L^2}{r^2} dr^2 + L^2 g_{ij} d\theta^i d\theta^j,$$

which describes the space $AdS_5 \times X_5$.

Type IIB theory on this background is expected to be dual to the IR limit of the field theory of N D3-branes at the conical singularity.

The simplest cones are orbifolds

$$Y_6 = R^6/\Gamma$$

Γ is a discrete subgroup of the $SO(6)$ rotation symmetry.

Can form 3 complex coordinates

$z^1 = x^1 + ix^2$; $z^2 = x^3 + ix^4$; $z^3 = x^5 + ix^6$

For $\Gamma \subset SU(3)$ that mixes the z's the $AdS_5 \times X^5$ is dual to $\mathcal{N}=1$ superconformal gauge theory.

The simplest representative of this class is the Z_3 orbifold generated by

$$\Gamma = e^{2\pi i/3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It produces a $SU(N) \times SU(N) \times SU(N)$ theory

Even more special are orbifolds that give $\mathcal{N}=2$ SCFT's.
The simplest example is $\mathbb{Z}_2$:

$$z^1 \to -z^1;\quad z^2 \to -z^2;\quad z^3 \to z^3 \quad (*)$$

In the gauge theory on D3-branes z^i are promoted to the 3 chiral superfields of the $\mathcal{N}=4$ SYM theory. The orbifold group also acts on the gauge indices.

Start with $2N$ D3-branes on the covering space $\Rightarrow U(2N)$ $\mathcal{N}=4$ SYM theory.

Keep fields invariant under $(*)$ together with conjugation by $\begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} I & \\ & -I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & \\ & -I \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$$

Find $U(N) \times U(N)$ gauge theory coupled to chiral superfields

A_1, A_2 in $(N, \bar{N})$

B_1, B_2 in $(\bar{N}, N)$

Φ in $(\text{adj}, 1)$

$\tilde{\Phi}$ in $(1, \text{adj})$

Superpotential

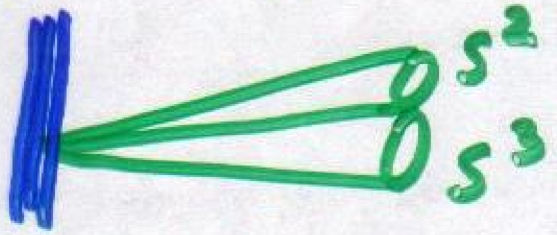
$\text{Tr}\, \Phi(A_1 B_1 - A_2 B_2) + \text{Tr}\, \tilde{\Phi}(B_1 A_1 - B_2 A_2)$

originates from $\text{Tr}\, Z^1 [Z^2, Z^3]$

in the parent $U(2N)$ $\mathcal{N}=4$ SYM.

$O(4) \sim SO(4) \times Z_2$ that rotates the z's.

An example of AdS/CFT is obtained by placing N D3-branes at the apex of the conifold



N D3-branes

The 10-d metric is

$$h^{-\frac{1}{2}}(r)\, dx_{11}^2 + h^{\frac{1}{2}}(r)\left(dr^2 + r^2 dS_{T^{11}}^2\right);$$

$$dx_{11}^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

In the near-horizon limit

$$h(r) = \frac{L^4}{r^4} \Rightarrow \text{find } AdS_5 \times T^{11}.$$

Type IIB string on $AdS_5 \times T^{11}$ is dual to the $\mathcal{N}=1$ SCFT on the N D3-branes

I. K., Witten
Morrison, Plesser

$SU(N) \times SU(N)$ gauge theory coupled to

A_1, A_2 in $(N, \bar{N})$

B_1, B_2 in $(\bar{N}, N)$

with superpotential

$$\epsilon^{ij} \epsilon^{kl} \mathrm{Tr}\,(A_i B_k A_j B_l)$$

The SCFT has global symmetry

$$U(1)_R \times U(1)_B \times SO(4) \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

The A's and B's have R-charge $\frac{1}{2}$.

$$SO(4) \sim \frac{SU(2) \times SU(2)}{\mathbb{Z}_2}$$

The $SU(2)$'s rotate A's and B's respectively

$U(1)_B: A_k \to e^{i\beta} A_k \,;\; B_k \to e^{-i\beta} B_k$

The massless gauge field in AdS_5 dual to the $U(1)_B$ global symmetry of the CFT is A_μ coming from

$\delta C_4 \sim A \wedge \omega_3$

(ω_3 is the harmonic 3-form on T^{11})

The discrete symmetry important for us is the $\mathbb{Z}_2$ that interchanges the A's and the B's and implements charge conjugation $N \leftrightarrow \bar{N}$ in each gauge group.

In the string theory this $\mathbb{Z}_2$ interchanges (θ_1, φ_1) and (θ_2, φ_2) of T^{11} accompanied by the center of the $SL(2,\mathbb{Z})$ S-duality symmetry of IIB strings.

Starting with the $\mathcal{N}=2$ $SU(N)\times SU(N)$ SCFT and adding relevant operator

$\mathrm{Tr}\,\Phi^2 - \mathrm{Tr}\,\tilde{\Phi}^2$

causes the flow to $\mathcal{N}=1$ SCFT whose superpotential is given by integrating out Φ and $\tilde{\Phi}$:

$$W \sim \mathrm{Tr}\,(A_1 B_1 - A_2 B_2)^2 - \mathrm{Tr}\,(B_1 A_1 - B_2 A_2)^2$$

$$\sim \epsilon^{ij}\epsilon^{kl}\,\mathrm{Tr}\,(A^i B^k A^j B^l)$$

which is $SU(2)\times SU(2)$ invariant!

Under this flow, the gravitational "central charge" (related to the $U(1)^3$ anomaly) decreases by a factor $\frac{27}{32}$ (consistent with the c-theorem).

This was checked by Gubser by both supergravity and field theory calculations.

Field theory interpretation of branes wrapped over the cycles of T^{11} was studied by IK, S. Gubser in hep-th/9808075. Since topologically $T^{11} \sim S^2 \times S^3$, we may wrap 3-branes over S^3.

On the field theory side the wrapped 3-branes correspond to the "dibaryon operators"

$$\epsilon_{\alpha_1 \dots \alpha_N} \epsilon^{\beta_1 \dots \beta_N} \prod_{i=1}^{N} A_{(k_i)\beta_i}^{\alpha_i} \quad (k_i = 1, 2) \text{ or analogously}$$

for the B's.

Calculating the volume of the minimal 3-cycle we find $\Delta = mL = T_3 V_3 L = \frac{3}{4} N$ in agreement with the fact that A's and B's have IR dimension $3/4$.

A D5-brane wrapped over a 2-cycle is a domain wall between $SU(N) \times SU(N)$ gauge theory and $SU(N) \times SU(N+1)$ gauge theory. This suggests a dual SUGRA approach to $SU(N_1) \times SU(N_2)$ theories.

CONIFOLD, DEFORMED OR RESOLVED.

$\vec{z} = \vec{x} + i\vec{y}$;

The singular conifold has $\vec{z}^{\,2} = 0 \Rightarrow$

$\vec{x}^{\,2} = \vec{y}^{\,2}$; $\vec{x}\cdot\vec{y} = 0$.

To find the base, impose also

$|\vec{z}|^2 = \rho^2 \Rightarrow \vec{x}^{\,2} + \vec{y}^{\,2} = \rho^2$;

$\vec{x}^{\,2} = \vec{y}^{\,2} = \frac{\rho^2}{2}$; $\vec{x}\cdot\vec{y} = 0$;

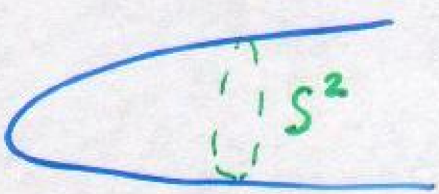
This defines an S^2 bundle over S^3 which is trivial.

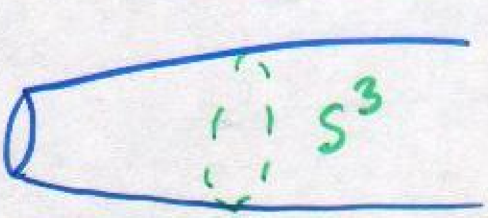
<u>Deformation</u>: $\vec{z}^{\,2} = \epsilon^2$;

Now $\vec{x}^{\,2} = \vec{y}^{\,2} + \epsilon^2$; $\vec{x}\cdot\vec{y} = 0$;

$\rho \geq \epsilon$ and at $\rho = \epsilon$ we find an

S^3: $\vec{x}^{\,2} = \epsilon^2$;





Resolution: introduce

$$z_{ij} = \frac{1}{\sqrt{2}} \begin{pmatrix} z_3 + i z_4 & z_1 + i z_2 \\ z_1 - i z_2 & -z_3 + i z_4 \end{pmatrix}$$

$\sum_{i=1}^{4} z_i^2 = 0$ implies $\det z_{ij} = 0$;

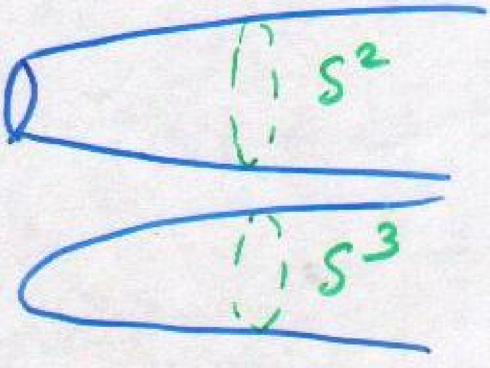
Introduce variables a_i, b_j; $i,j = 1,2$

$$z_{ij} = a_i b_j \; ; \quad (a_i, b_j) \sim (e^{i\theta} a_i, e^{-i\theta} b_j)$$

$$\sum_i |a_i|^2 - \sum_j |b_j|^2 = \delta \; ;$$

For $\delta = 0$ we find the singular conifold: the base is now described as $\frac{S^3 \times S^3}{U(1)}$.

For $\delta > 0$ we find a resolved conifold when $\sum_j |b_j|^2 = 0$ (the S^3 shrinks) we are left with $S^3/U(1) \equiv S^2$.



Gauge invariant operators correspond to modes on $AdS_5 \times X_5$. The correlation functions may be calculated from the GKPW formula:

$$e^{-I(\phi_0)} = \left\langle e^{\int d^d x\, \phi_0(x)\, \mathcal{O}(x)} \right\rangle_{\text{gauge theory.}}$$

$I(\phi_0)$ is the SUGRA action as a function of the boundary behavior on AdS_{d+1};

$$\phi(z, x) \to z^{d-\Delta}\left[\phi_0(x) + \mathcal{O}(z^2)\right] + z^{\Delta}\left[A(x) + \mathcal{O}(z^2)\right],$$

$$ds^2_{AdS} = \frac{1}{z^2}\left(dz^2 + d\vec{x}^{\,2}\right), \quad \text{(we set } L = 1\text{)}.$$

ϕ is the scalar of mass m, and the corresponding operator dimension Δ is determined from

$$\Delta(\Delta - d) = m^2 L^2; \Rightarrow \Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2 L^2};$$

For $L^2 m^2 > -\frac{d^2}{4} + 1$ we must choose Δ_+, but in the Breitenlohner-Freedman range

$$-\frac{d^2}{4} < m^2 L^2 < -\frac{d^2}{4} + 1$$

both Δ_+ and Δ_- are allowed. The lower bound on Δ is $\frac{d}{2} - 1$, the unitarity bound in CFT_d (Witten, IK).

The simplest class of chiral operators in the conifold field theory are

$$tr\,(A^{i_1}B_{j_1}A^{i_2}B_{j_2}\ldots A^{i_k}B_{j_k})$$

where the i's and j's are separately symmetric.
The R-charge is k and $SU(2)\times SU(2)$ quantum numbers are $(\frac{k}{2},\frac{k}{2})$.

Since $\Delta=\frac{3k}{2}$, for $k=1$ we have $\Delta<\frac{d}{2}$.
Corresponding scalar in AdS_5 has $m^2=-\frac{15}{4}$ (conformally coupled)

$$-4<m^2<-3$$

SUSY requires that we pick $\Delta_-=2-\sqrt{4+m^2}=\frac{3}{2}$ for this operator.
Had we picked Δ_+ the theory on AdS_5 would not be supersymmetric.

Im

The two gauge couplings are related to the dilaton and NS-NS 2-form by

$$\frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2} = \frac{1}{g_s} e^{-\phi} ;$$

$$\frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_2^2} = \frac{1}{g_s} e^{-\phi} \left[\frac{1}{2\pi\alpha'} \int_{S^2} B_2 - \pi \right] .$$

This $SU(N) \times SU(N)$ SCFT exists for any N, g_1, g_2.

$AdS_5 \times T_{11}$ should be a soln of full IIB string, not just SUGRA.

We may check the leading higher-derivative correction (Frolov, Ik, Tseytlin).

$$\delta S = \frac{1}{16\kappa^2} \zeta(3) \alpha'^3 \int d^{10}x \sqrt{g}\, e^{-3/2\,\phi} W ,$$

$$W = C^{hmnk} C_{pmnq} C_h{}^{rsp} C^q{}_{rsk} + \frac{1}{2} C^{hkmn} C_{pqmn} C_h{}^{rsp} C^q{}_{rsk} .$$

For $AdS_5 \times T^{11}$, the Weyl tensor $C^a{}_{bcd} \neq 0$.

But $W = 0$ and $\frac{\partial W}{\partial g_{ab}} = 0$. ✓

Topologically, $T^{11} \sim S^2 \times S^3$.

If we add M D5-branes wrapped over the S^2 to the N D3-branes at the singularity, then the gauge group changes to $SU(N+M) \times SU(N)$. (Gubser, IK).

To implement the SUGRA dual of this non-conformal $\mathcal{N}=1$ gauge theory, we add M units of the 3-form R-R flux

$$\int_{S^3} H^{RR} \propto M$$ (produced by the D5's)

to the N units of 5-form flux produced by the regular D3-branes

$$\int_{T^{11}} F_5 \propto N.$$

H^{RR} induces the radial dependence of

$\int_{S^2} B^{NS-NS}$, which translates into flow of $\frac{1}{g_1^2} - \frac{1}{g_2^2}$

(Nekrasov, IK)

The Einstein metric on T^{11} is

$$ds^2_{T^{11}} = \frac{1}{9}\left(d\psi + \cos\theta_1 d\varphi_1 + \cos\theta_2 d\varphi_2\right)^2 + $$
$$+ \frac{1}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2\right)$$

$\psi \in [0, 4\pi)$; $\varphi_1, \varphi_2 \in [0, 2\pi)$; $\theta_1, \theta_2 \in [0, \pi)$

The harmonic 2- and 3-forms are

$$\omega_2 = \frac{1}{2}\left(\sin\theta_1 d\theta_1 \wedge d\varphi_1 - \sin\theta_2 d\theta_2 \wedge d\varphi_2\right),$$

$$\omega_3 = e^\psi \wedge \omega_2,$$

$$e^\psi = d\psi + \cos\theta_1 d\varphi_1 + \cos\theta_2 d\varphi_2$$

$$\int \omega_2 = 4\pi \;; \qquad \int \omega_3 = 8\pi^2;$$

To have M units of RR 3-form flux, we need $\frac{1}{4\pi^2\alpha'}\int F_3 = M$;

The solution with this extra flux through T^{11} is

I.K., Tseytlin

$F_3 = \frac{\alpha'}{2} M\, \omega_3\,; \qquad B_2 = \frac{\alpha'}{2} 3 g_s M \ln(r/r_0)\, \omega_2$

The dilaton $\Phi = 0$.

$$ds^2 = h^{-\frac{1}{2}}(r)\, dx_\mu dx^\mu + h^{\frac{1}{2}}(r)\left(dr^2 + r^2 ds^2_{T^{11}}\right)$$

$$h(r) = \frac{4\pi g_s}{r^4}\left(N_0 + \frac{3}{2\pi} g_s M^2 \left[\ln\left(\frac{r}{r_0}\right) + \frac{1}{4}\right]\right)\frac{27}{16}$$

The 5-form field strength $\tilde{F}_5 = \mathcal{F}_5 + * \mathcal{F}_5$,

$\mathcal{F}_5 \sim N(r)\ \mathrm{vol}\,(T^{11})$.

$$N(r) = N_0 + \frac{3}{2\pi} g_s M^2 \ln(r/r_0).$$

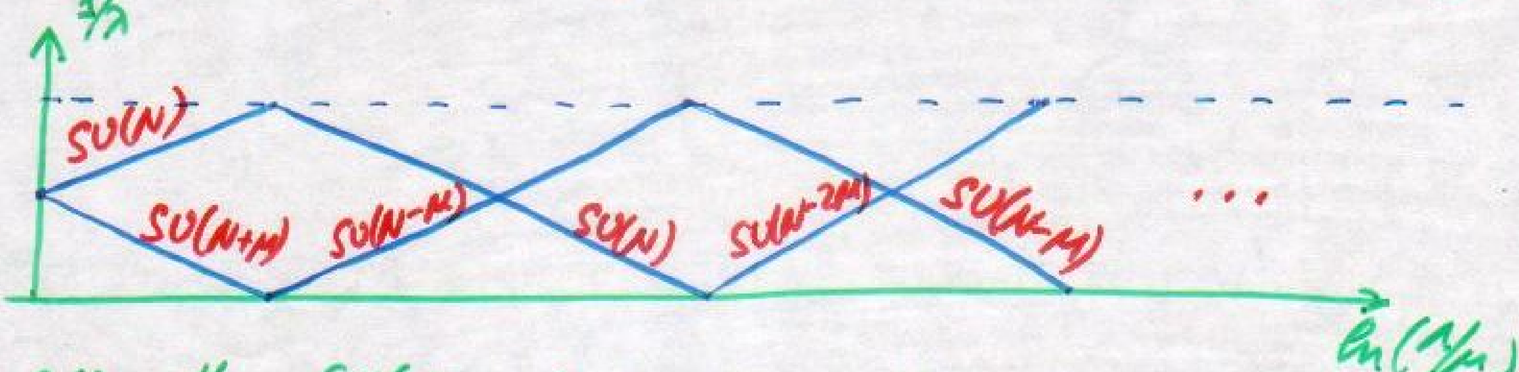
The number of colors N in $SU(N) \times SU(N+M)$ has become scale dependent.

As $\ln(r/r_0)$ decreases by $\frac{2\pi}{3 g_s M}$,

$N(r) \to N(r) - M$.

This is the cascade of Seiberg dualities.

On the FT side the jumps are due to the $\mathcal{N}=1$ Seiberg duality (IK, Strassler; in progress).



After the SU(N+M) coupling diverges, application of duality gives $SU(N_f - N_c) = SU(N-M)$ group.

Suppose that $N = 10^6 M + p$; $p \sim O(1)$.

After 10^6 jumps we reach $SU(p) \times SU(M+p)$, and further jumps are impossible.

If $g_s M \gg 1$ then the SUGRA background

$$h(r) \sim \frac{(g_s M)^2 \ln(r/r_*)}{r^4}$$

appears to have small curvatures.

Note also the appearance of the SU(M) 't Hooft coupling $g_s M$.

However, there is a naked singularity at $r = r_*$.

Connection with the NSVZ β-functions.

$$\frac{d}{d\ln(\Lambda/\mu)}\frac{8\pi^2}{g_1^2} \sim 3(N+M) - 2N(1-\gamma)$$

$$\frac{d}{d\ln(\Lambda/\mu)} \cdot \frac{8\pi^2}{g_2^2} \sim 3N - 2(N+M)(1-\gamma)$$

γ is the anomalous dimension of $\mathrm{Tr}(A_i B_j)$

$$\gamma = -\frac{1}{2} + \mathcal{O}\left(\frac{M^2}{N^2}\right).$$

$$\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} \sim 6M \ln(\Lambda/\mu)$$

In SUGRA we have

$$\int B_2 = (2\pi\alpha')\, 3 g_s M \ln(r/r_0)$$

Using the relation

$$\frac{1}{2\pi\alpha' g_s}\int B_2 = \frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_2^2}$$

we find exact agreement with FT.
The factor 3 in the SUGRA calculation is geometrical; related to self-duality of G_3.
Correct β-function also follows from $W \sim \int G_3 \wedge \Omega$
Vafa.

Since the A's and the B's have R-charge $\frac{1}{2}$, the $U(1)_R$ anomaly eqn. of the $SU(N)\times SU(N+M)$ is

$$\partial_i J^i = \frac{M}{16\pi^2}\left(F^a_{ij}\tilde{F}^{aij} - G^b_{ij}\tilde{G}^{bij}\right)$$

How to derive this from dual SUGRA?

IK, P. Ouyang, E. Witten

Consider a transformation $\frac{\psi}{2} \to \frac{\psi}{2} + \epsilon(x^i)$.

$C_2 \to C_2 + M\alpha'\epsilon\,\omega_2$;

$\theta_1 - \theta_2 = \frac{1}{\pi\alpha'}\int_{S^2} C_2$; $\quad \theta_1 + \theta_2 \sim C = 0$

The $U(1)_R$ rotation generates $\theta_1 = -\theta_2 = 2M\epsilon$.

The terms linear in ϵ in the dual gauge theory are

$$\int d^4x\left[-\epsilon\,\partial_i J^i + \frac{M\epsilon}{16\pi^2}\left(F^a_{ij}\tilde{F}^{aij} - G^b_{ij}\tilde{G}^{bij}\right)\right]$$

Vanishing reproduces the anomaly eqn.

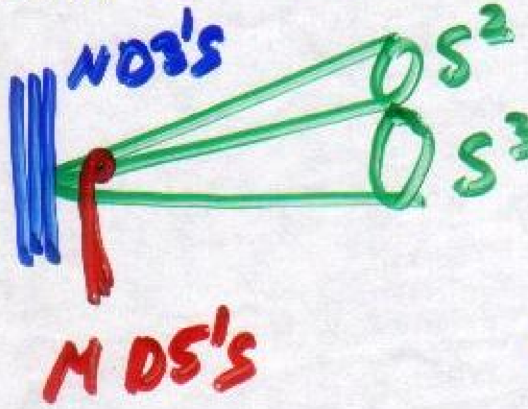
Field theory periodic under $\theta \to \theta + 2\pi k$;

i.e. $\psi \to \psi + \frac{2\pi k}{M}$; since $\psi \sim \psi + 4\pi$,

Z_{2M} is the symmetry unbroken by anomaly.

THE WARPED DEFORMED CONIFOLD

Add M D5-branes wrapped over the S^2 at the apex.



This adds M units of RR 3-form flux through the S^3, which causes a geometric transition to the deformed conifold

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = \epsilon^2$$

IK, Strassler
Vafa
Maldacena, Nunez

The warped 10-d metric is

$$h^{-\frac{1}{2}}(\tau)\, dx_{\parallel}^2 + h^{\frac{1}{2}}(\tau)\, ds_6^2$$

The CY metric of the deformed conifold is known explicitly

$$ds_6^2 = \frac{\epsilon^{4/3} K(\tau)}{2}\Bigg[\frac{1}{3K^3}\left(d\tau^2 + g_5^2\right)$$

$$+ \cosh^2\left(\frac{\tau}{2}\right)\left(g_3^2 + g_4^2\right) + \sinh^2\left(\frac{\tau}{2}\right)\left(g_1^2 + g_2^2\right)\Bigg]$$

$$g^1 = \frac{e^1 - e^3}{\sqrt{2}}\,; \quad g^2 = \frac{e^2 - e^4}{\sqrt{2}}\,;$$

$$g^3 = \frac{e^1 + e^3}{\sqrt{2}}\,, \quad g^4 = \frac{e^2 + e^4}{\sqrt{2}}\,; \quad g^5 = e^5\,;$$

$$e^5 = d\psi + \cos\theta_1\, d\varphi_1 + \cos\theta_2\, d\varphi_2\,;$$

$$e^1 = -\sin\theta_1\, d\varphi_1\,; \quad e^2 = d\theta_1\,;$$

$$e^3 = \cos\psi \sin\theta_2\, d\varphi_2 - \sin\psi\, d\theta_2\,;$$

$$e^4 = \sin\psi \sin\theta_2\, d\varphi_2 + \cos\psi\, d\theta_2\,.$$

$$K(\tau) = \frac{\left(\sin(2\tau) - 2\tau\right)^{1/3}}{2^{1/3}\,\sinh\tau}\,.$$

At $\tau = 0$ the S^2 shrinks, and a round S^3 is left.

The warp factor is monotonic and reaches a finite value at $\tau=0$:

$$h(\tau) = (g_s M\alpha')^2 \, 2^{2/3} \epsilon^{-8/3} \int_\tau^\infty dx \, \frac{x\coth x - 1}{\sinh^2 x} (\sinh(2x) - 2x)^{1/3}$$

Holographic calculation of Wilson loops gives AREA LAW.

The background also includes complex 3-form of type (2,1) and the self-dual 5-form field strength.

The dual gauge theory is the cascading $SU(N+M) \times SU(N)$ coupled to A_1, A_2 in $(N+M, \bar{N})$, B_1, B_2 in $(\overline{N+M}, N)$.

The conifold metric is

$$dr^2+r^2\left\{\frac{1}{9}(g^5)^2+\frac{1}{6}\sum_{i=1}^{4}(g^i)^2\right\}.$$

The deformed conifold metric is also diagonal:

$$ds_6^2=\frac{1}{2}\epsilon^{\frac{4}{3}}K(\tau)\Big[\frac{1}{3K^3(\tau)}\left(d\tau^2+(g^5)^2\right)+$$

$$+\cosh^2\left(\frac{\tau}{2}\right)\left[(g^3)^2+(g^4)^2\right]+\sinh^2\left(\frac{\tau}{2}\right)\left[(g^1)^2+(g^2)^2\right]\Big]$$

$$K^3(\tau)=\frac{\sinh(2\tau)-2\tau}{2\sinh^3\tau}$$

At $\tau=0$ the angular metric degenerates to

$$d\Omega_3^2\sim\frac{1}{2}(g^5)^2+(g^3)^2+(g^4)^2,$$

which is a round 3-sphere.

The 3-form ansatz is

$$F_3=M\left\{g^5\wedge g^3\wedge g^4+d\left[F(\tau)(g^1\wedge g^3+g^2\wedge g^4)\right]\right\}$$

$$B_2=g_sM\left\{f(\tau)\,g^1\wedge g^2+k(\tau)\,g^3\wedge g^4\right\}$$

and $\mathcal{F}_5=B_2\wedge F_3=g_sM^2\,\ell(\tau)\,g^1\wedge g^2\wedge g^3\wedge g^4\wedge g^5$

$\ell(\tau)=f(1-F)+kF$

The 10-d metric ansatz is a

WARPED DEFORMED CONIFOLD:

$$ds_{10}^2 = h^{-\frac{1}{2}}(\tau)\, dx_{\parallel}^2 + h^{\frac{1}{2}}(\tau)\, ds_6^2 .$$

The 1-st order equations that reproduce all the 2-nd order IIB SUGRA equations are

$$\left.\begin{aligned} f' &= (1-F)\tanh^2(\tau/2) \\ k' &= F \coth^2(\tau/2) \\ F' &= \frac{1}{2}(k-f) \end{aligned}\right\}$$

define a self-dual 3-form

$$g_s *_6 F_3 = -H_3$$

$$g_s F_3 = *_6 H_3$$

Complex 3-form

$$G_3 = F_3 + \frac{i}{g_s} H_3$$

is a $(2,1)$ form on the deformed conifold.

$$h' = -\alpha \frac{f(1-F)+kF}{K^2(\tau)\sinh^2\tau}$$

$$\alpha \sim (g_s M)^2$$

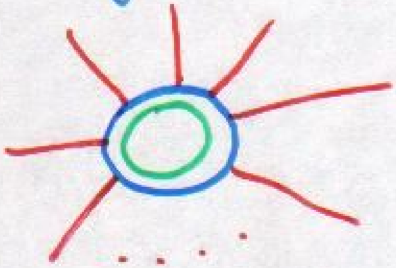
Φ = constant since $H_3^2 = F_3^2$.

The explicit solution which approaches the KT (conifold) solution for large τ is

$$\begin{matrix} f(\tau) \\ k(\tau) \end{matrix} = \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau \mp 1); \quad F(\tau) = \frac{\sinh\tau - \tau}{2\sinh\tau};$$

The presence of a finite S^3 far in the IR (at $\tau=0$) accounts for much of the "phenomenology" of a confining $\mathcal{N}=1$ theory.

1. A D3-brane wrapped over the S^3 is a BARYON VERTEX connecting M strings (external quarks).



2. A D5-brane wrapped over the S^3 is a DOMAIN WALL in $R^{3,1}$ separating two adjacent vacua.

For large $g_s M$ curvatures are small everywhere $\Rightarrow$ we find a reliable SUGRA dual of a cascading, confining theory.

$\delta C_2 = \pi \alpha' \omega_2$;

C_2 can be locally written as $M \frac{\alpha'}{2} \psi \omega_2$.

Upon crossing the wall, $\psi \to \psi + \frac{2\pi}{M}$.

Using the wrapped D5-branes we can indeed generate M inequivalent orientations in ψ.

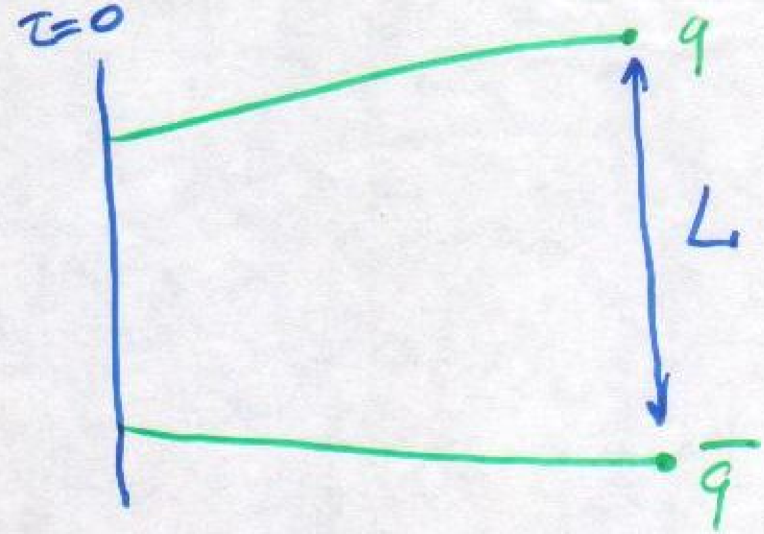
$\langle \lambda\lambda \rangle$ may be explicitly found holographically (Loewy, Sonnenschein)

Calculate the large-τ correction to the KT form of $C_2 + \frac{i}{g_s} B_2$.

It is $-M \frac{\tau}{2} e^{-\tau} e^{-i\psi} (d\theta_1 + i \sin\theta_1 d\varphi_1) \wedge$

$(d\theta_2 + i \sin\theta_2 d\varphi_2)$.

Since $\tau e^{-\tau} \sim \frac{\ln r}{r^3}$, this corresponds to expectation of dimension 3 operator (up to logarithmic corrections).



The $q\bar{q}$ potential is linear since most of the string is near $\tau=0$ where the longitudinal metric is $\sim \dfrac{\epsilon^{4/3}}{g_s M (\alpha')^2} dx_{\parallel}^2$.

The spectrum of glueballs (normalizable wave functions) on the entire smooth background is DISCRETE (Caceres, Hernandéz; Krasnitz)

Their masses $\sim \dfrac{\epsilon^{2/3}}{g_s M \alpha'}$

For large τ, $\tau = 3 \ln(r/\epsilon^{2/3}) + \text{const}$

$\epsilon^{2/3}$ emerges through DIMENSIONAL TRANSMUT.

A fundamental string at the apex $(r=0)$ along the world volume directions $\mathbb{R}^{3,1}$ has tension $\sim \frac{1}{\sqrt{h(0)}}$.

It is finite $\Rightarrow$ CONFINEMENT

In contrast, for $AdS_5 \times S^5$, $h = \frac{L^4}{r^4}$ blows up at $r=0$ $\Rightarrow$

NO CONFINEMENT FOR $\mathcal{N}=4$ SUPER-YM.

<u>IN THE $\mathcal{N}=1$ $SU(M)$ GAUGE THEORY</u>
<u>THERE ARE CONFINING q-STRINGS</u>

anti-quarks :  : quarks $\Leftarrow$ a 2-string

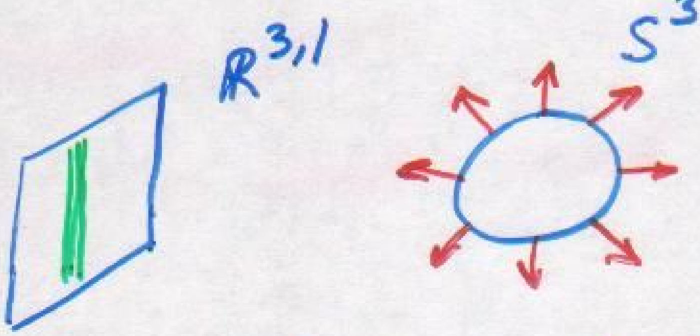
They connect q probe quarks with q corresponding anti-quarks.

Wilson loops in the fully antisymmetric representation with q indices.

The tensions T_q, $q=1,2,\dots,M-1$ ..

$T_q = T_{M-q}$ ($q \to M-q$ exchanges quarks with antiquarks).

$T_M = 0$ (M quarks form a colorless object, the baryon).



In our SUGRA dual the q-string is described by q coincident fundamental strings at the apex.

The blown-up 3-sphere has metric ($b \simeq .933$)

$$ds_3^2 = b\, g_s M \alpha' \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right].$$

and M units of R-R 3-form flux

$$F_3 = 2M\alpha' \sin^2\psi \sin\theta\, d\psi \wedge d\theta \wedge d\varphi.$$

$\psi \in [0, \pi]$ is the azimuthal angle.

The R-R flux blows up the q F-strings into a D3-brane wrapped over an S^2 at fixed azimuthal angle ψ C. Herzog, I.K.

$$T \sim \sqrt{b^2 \sin^4\psi + \left(\psi - \frac{\sin 2\psi}{2} - \pi\frac{q}{M}\right)^2}$$

Minimizing wrt ψ, we find

$$\psi - \frac{\pi q}{M} = \frac{1-b^2}{2}\sin(2\psi)$$

If $b=1$, then $\psi = \frac{\pi q}{M}$

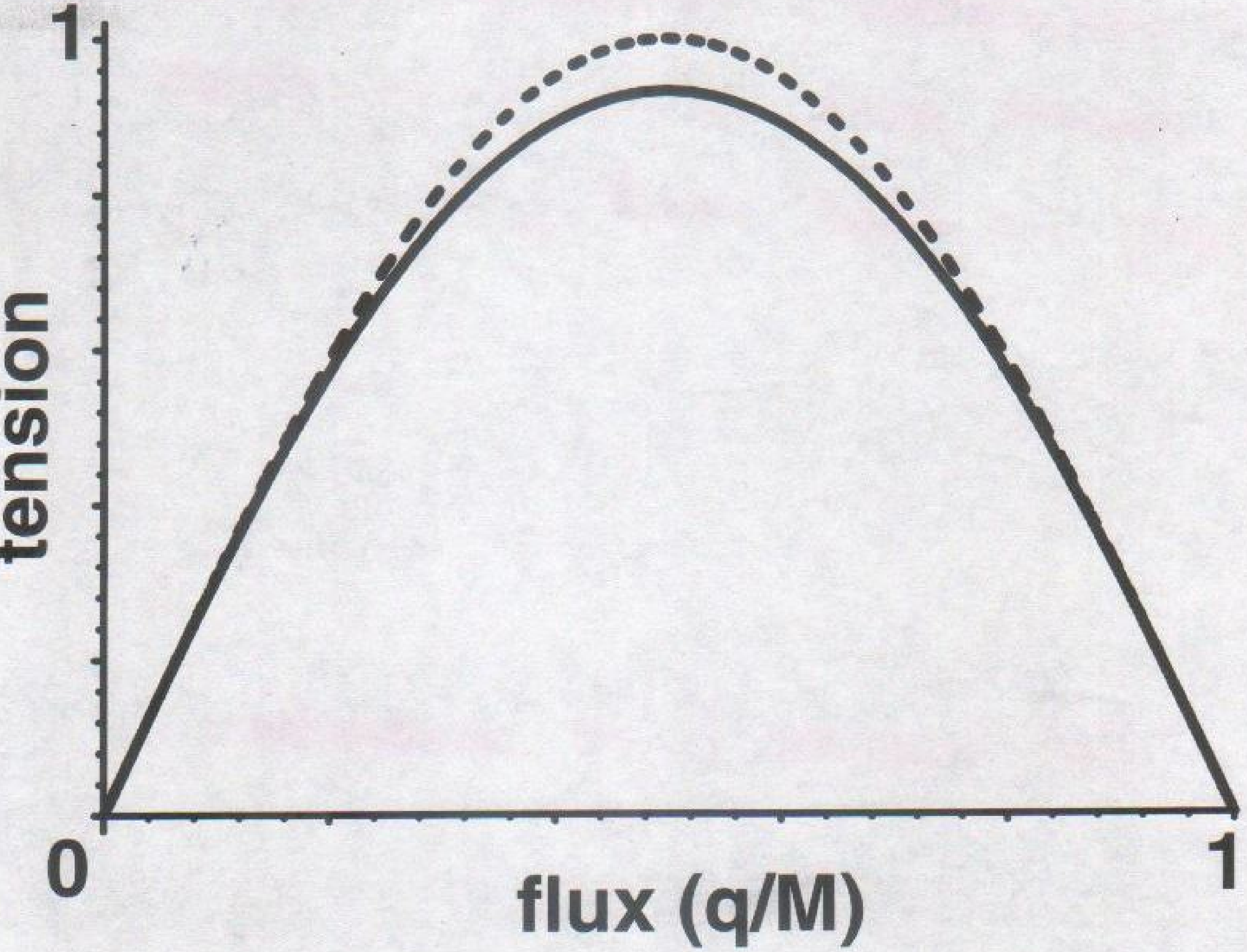
$$T_q \sim \sin\left(\pi\frac{q}{M}\right)$$

This formula is valid in softly broken $\mathcal{N}=2$ and in MQCD.

For the KS solution, $b \approx .933 \Rightarrow T_q$ is almost the same.

For the Maldacena-Nuñez SUGRA dual, which also has a blown-up S^3 $b=1$ exactly.

q-string tensions



Now the $U(1)_R$ is broken by the chiral anomaly to Z_{2M} in the UV.

The holographic dual of this is understood

IK, Ouyang, Witten.

Furthermore, in the IR

$Z_{2M} \to Z_2$ by the gluino condensate.

The remaining Z_2 symmetry is

$z_k \to -z_k$ symmetry of the deformed conifold.

The $SO(4)$ rotational symmetry is unbroken. There is also a Z_2 that extends it to $O(4)$: $z_4 \to -z_4$, accompanied by $G_3 \to -G_3$ (the $-I$ of $SL(2,Z)$)

In the gauge theory this is $A_k \leftrightarrow B_k$ accompanied by charge conjugation.

What is the fate of the $U(1)_B$?

It is not anomalous in the UV.

In the IR it is spontaneously broken by expectation values of baryonic operators.

IK Strassler

In the $SU(2M) \times SU(M)$ gauge theory at the bottom of the cascade, there is superpotential

$$X[\det N - \mathcal{B}\tilde{\mathcal{B}} - \Lambda^{4M}]$$

$N \sim AB$ is the meson matrix.

Baryon operators exist because the $SU(2M)$ gauge theory effectively has $2M$ flavors:

$$\mathcal{B} \sim A_1^M A_2^M \; ; \quad \tilde{\mathcal{B}} \sim B_1^M B_2^M$$

On the BARYONIC BRANCH Argyres Plesser Seiberg

$X=0;\ N=0;\ \mathcal{B}\tilde{\mathcal{B}}=-\Lambda^{YM}$

the SO(4) global symmetry is unbroken => this is a good candidate for the dual of the warped deformed conifold. IK, Strassler

But where is the pseudoscalar Goldstone boson of the spontaneously broken $U(1)_B$? Aharony.

A seemingly unrelated old puzzle is: what is the gauge theory dual of the D-string at $\tau=0$?

Just as the F-string, dual to the confining string, the D-string has

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a non-vanishing tension at $\tau = 0$.
What is it in the gauge theory?
The two puzzles have a common solution! Gubser, Herzog, Ik

The D-string is dual to the axionic string in the gauge theory.
The axion ≡ pseudoscalar Goldstone boson of the spontaneously broken $U(1)_B$ symmetry.
This normalizable mode around the warped deformed conifold background, i.e. a massless glueball, comes from the RR-sector.

THE VARIATIONS

$$\delta F_3 = *_4 da - d\left[f_2(\tau)\, da \wedge g^5\right]$$

$$\delta F_5 = \left(*_4 da - \frac{\epsilon^{4/3} h(\tau)}{6K^2(\tau)}\, da \wedge d\tau \wedge g^5\right) \wedge B_2$$

$$B_2 = \frac{g_s M \alpha' (\tau \coth \tau - 1)}{2\, \mathrm{sh}\, \tau}\left[\mathrm{sh}^2\left(\frac{\tau}{2}\right) g^1 \wedge g^2 + \mathrm{ch}^2\left(\frac{\tau}{2}\right) g^3 \wedge g^4\right]$$

is the background NS-NS field, and

$\delta B_2 = 0$.

Linearized SUGRA equations require $a(x^0, x^1, x^2, x^3)$ to be harmonic:

$$d *_4 da = 0.$$

This is the 4-d axion mode.

The equation for f_2 has a normalizable solution

$$f_2 \sim \frac{1}{K^2\, \mathrm{sh}^2 \tau} \int_0^\tau h(x)\, \mathrm{sh}^2 x\, dx$$

For large τ, $f_2 \sim \tau e^{-2\tau/3}$

For small τ, $f_2 \sim \tau$

The form of δF_5 shows that da is the longitudinal component of the vector field dual to the baryon current. Hence, the 4-d effective action contains

$$\frac{1}{f_a} \int d^4x \, J^\mu \partial_\mu a$$

On the baryonic branch

$$\mathcal{B} = i \xi \Lambda^{2M}; \quad \tilde{\mathcal{B}} = \frac{i}{\xi} \Lambda^{2M},$$

a enters as the phase of ξ.

SUSY also requires the existence of a modulus field, "a saxion".

Just like the pseudoscalar mode, this scalar breaks the Z_2 symmetry of the background ($(\theta_1, \varphi_1) \leftrightarrow (\theta_2, \varphi_2)$, together with $G_3 \to -G_3$).

$\delta B_2 = \chi(\tau) \, d g^5$;

$\delta G_{13} = \delta G_{24} = m(\tau)$

$g^1 g^3 + g^3 g^1 + g^2 g^4 + g^4 g^2 = \sin\theta_1 \, d\theta_1 \, d\varphi_1 - \sin\theta_2 \, d\theta_2 \, d\varphi_2$

are the same metric components turned on in a SMALL RESOLUTION of the conifold. Pando Zayas, Tseytlin

Solving the SUGRA eqns. we find

$m(\tau) \sim h^{1/2}(\tau) \left(\tau \coth\tau - 1 \right)$

and $\chi(\tau)$ is determined by $m(\tau)$.

COMPACTIFICATION

The warped deformed conifold throat may be embedded into a CY compactification with fluxes. GKP

Then global symmetries of the gauge theory become gauged.

On the baryonic branch, we then find SUSY Higgs mechanism for the $U(1)_B$. The pseudoscalar gets "eaten" and becomes part of massive vector field; the scalar Higgs is degenerate with it — they belong to a $\mathcal{N}=1$ massive vector supermultiplet.

Now we expect D-strings to be dual to the Abrikosov-Nielsen-Olesen strings.

Such strings are not expected to be BPS. Since there are K units of NS-NS 3-form flux, K parallel D-strings can "break" on a wrapped D3-brane.

Hence, D-string charge should take values in $\mathbb{Z}_K$.

Copeland
Myers
Polchinski

CONCLUSIONS

The warped deformed conifold background of type IIB string theory is dual to the cascading gauge theory on the baryonic branch in the Z_2 symmetric case $|\mathcal{B}| = |\tilde{\mathcal{B}}|$

Transformation $\mathcal{B} \to (1+\alpha)\mathcal{B}$; $\tilde{\mathcal{B}} \to \frac{\tilde{\mathcal{B}}}{1+\alpha}$

breaks the Z_2; its SUGRA dual is understood to linear order in α.

Finding exact SUGRA solutions dual to the entire baryonic branch,

THE RESOLVED WARPED DEFORMED CONIFOLDS,

remains a challenge.