Nonradial Oscillations on Accreting Neutron Stars

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A theoretical study of g-modes on accreting neutron stars, including stability tests and the effects of rotation.
Why Nonradial Oscillations?

- Non-spherically symmetric processes are known to be occurring on accreting neutron stars
- Oscillations are seen from some accreting neutron stars (NOT the same as models discussed here)

Hasn’t this been done before?

- McDermott et al. 1985
- McDermott & Taam 1987
- Strohmayer & Lee 1996

Key Differences

- Thermally stable envelope models
- Rotational effects
Thermal Instability

- Observed as type I X-ray bursts (Grindlay et al. 1976; Belian, Conner & Evans 1976)
- Take constant pressure perturbation of nuclear energy generation rate & radiative cooling equations (Hanawa & Fujimoto 1982; Fushiki & Lamb 1987)
- Steady state burning when \( \varepsilon_{3\alpha} < T^\gamma \rho^2, \nu = \frac{4\gamma}{T_\gamma} - 3 \)
  \[ \nu - 4 + \frac{d\ln K}{d\ln T} + \frac{d\ln \rho}{d\ln T} (2 + \frac{d\ln K}{d\ln \rho}) < 0 \]

\[ T \geq 4.8 \times 10^8 K \] for stability

(Review in Bildsten 1998)

KEY: Thermal stability has a local criteria
Steady State Burning Envelope

- Pure Helium accretion for simplicity

- Local Eddington rate

\[ m_{\text{edd}} = \frac{2m_{\text{pc}}}{R \sigma_{\text{tr}}} = 1.5 \times 10^{-5} \frac{\text{g}}{\text{cm}^2 \text{s}} \left( \frac{10 \text{ km}}{R} \right) \]

- \( h\ll R \\Rightarrow \) plane parallel geometry

\[ g \sim 2 \times 10^{14} \frac{\text{cm}}{s^2} \left( \frac{10 \text{ km}}{R} \right)^2 \left( \frac{M}{1.4 M_\odot} \right) \]

- Hydrostatic balance \( P = g \gamma_z, \ dy = -pdz \)

3 Differential Equations (Bildsten 1995)

\[
\begin{align*}
\frac{dT}{dy} &= \frac{8KF}{4acT^4} \\
\frac{dY}{dy} &= -\frac{12mpE_3\alpha}{C_3\alpha m} \\
\frac{dF}{dy} &= -E_3\alpha
\end{align*}
\]

(Pacyenski 1983 fn. E.O.S.)
Envelope Profiles vs. Accretion Rate

Rates = 1, 5, 10, and $20 \times \dot{m}_{\text{Edd}}$

Brunt-Väisälä frequency

$$N^2 = \frac{g}{h} \left[ \frac{X_T}{X_P} \left( \frac{d \ln T}{d \ln P} - \frac{d \ln T}{d \ln P_\star} \right) - \frac{X_{\mu i}}{X_P} \frac{d \ln \mu_i}{d \ln P_\star} \right]$$

[Graph showing the relationship between Y, N (10^4 Hz), T (10^8 K), and column density (g cm^-2) with increasing accretion rate indicated.]
Envelope Profile vs. Base Flux

All with accretion rate = \( \dot{m}_{\text{edd}} \)

\[ F_{\text{base}} = 0.0, 0.5, 1.0, \text{ and } 1.5 \times F_{\text{burn}} \]

\[ F_{\text{burn}} = \frac{Q_{32} \dot{m}}{12 m_p} = 8.76 \times 10^{22} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ (m}_{\text{edd}}) \]

\[
\begin{array}{c|c|c|c|c|c}
N (10^4 \text{ Hz}) & T (10^8 \text{ K}) & \text{Y} & \text{increasing} & \text{base flux} \\
\hline
10^7 & 10^8 & 10^9 & \text{g cm}^{-2} &
\end{array}
\]
Nonradial Mode Equations

- $t_{\text{thermal}} \gg t_{\text{dynamic}} \Rightarrow$ adiabatic approx.
- Start with equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla P - \rho \nabla \Phi$$

- Introduce Eulerian perturbations

$P \rightarrow P_0 + \delta P$, $\rho \rightarrow \rho_0 + \delta \rho$, $\mathbf{v} \rightarrow \delta \mathbf{v} = \frac{d \xi}{d t}$, etc.

- Linearize result with assumptions

$\delta Q = \delta Q(r, \theta) e^{i m \theta + i \omega t}$ and $\frac{\Delta \rho}{\rho} = \frac{i}{\Pi} \frac{\Delta P}{P}$

**Resulting Equations**

$$\frac{d}{dr} \frac{\delta P}{P} = \left( 1 - \frac{1}{\Pi} \right) \frac{\delta P}{P} + \left( \frac{\omega^2}{g} - \frac{N^2}{g} \right) \frac{\delta r}{h}$$

$$\frac{d}{dr} \frac{\delta r}{r} = \frac{\delta r}{h} + \left( \frac{g \ell (\ell + 1)}{w^2 R^2} - \frac{1}{\Pi} \right) \frac{\delta P}{P}$$

$h = P/\rho g = \text{scale height}$, $N^2 = \text{Brunt-Väisälä freq.}$
Radial Eigenfunctions

- Top boundary condition
  \[ \Delta p = 0 \text{ at } \theta_{\text{thermal}} = \Pi \] \( (y \approx 10^5 \text{ g cm}^{-2}) \)

- Bottom boundary condition
  \[ \frac{\partial}{\partial r} \xi_r = 0 \text{ at iron layer } (y \approx 10^{12} \text{ g cm}^{-2}, \text{ Bildsten } '98) \]

\[ |\xi_1| = \frac{gh_k}{\omega^2} \left| \frac{SP}{P} \right| \xi_1 \frac{dE}{d\ln y} = 2\pi R^2 \omega^2 \xi_1 \frac{dE}{d\ln y} \]

![Graph showing log-log plot of \( \xi_r \) and \( \xi_1 \) against column depth]

\[ \text{Column Depth (g/cm}^2) \]

\[ \text{Log}(\text{dE/dlny}) \]
More Radial Eigenfunctions...

- Flatter energy density profiles

\[ |\xi_1| = \frac{8\hbar^2}{\omega^2} \left| \frac{\delta P}{P} \right| \xi_1 \frac{dE}{dlny} = 2\pi R^2 \omega^2 \xi_1^2 \gamma \]

\[ \log_{10}|\xi_1|, \log_{10}(dE/dlny) \]

\[ 17 \text{ Hz}, 13 \text{ Hz} \]

Column Depth (g/cm²)
G-Mode Frequency vs. Accretion Rate

(Base Flux = 0)

From WKB we expect

\[ f \propto h^{1/2} \propto T^{1/2} \propto F^{1/8} \propto m^{1/8} \]

2nd, 3rd, 4th, and 5th G-Modes

![Graph showing G-Mode Frequency vs. Accretion Rate](image_url)
**G-Mode Frequency vs. Base Flux**

(Accretion Rate = $\dot{m}/Edd$)

$F_{\text{burn}} \Rightarrow 0.6 \text{ MeV/nucleon} \times \dot{m}$

More shallow mode is less sensitive to bottom boundary condition.
Mode Stability (Cox 1980; Unno et al. '89)

- We want to understand exchange of energy between mode and star.

- In adiabatic case we can define a linear Hermitian operator \( \mathcal{L} \),
  \[
  \mathcal{L}(\vec{\xi}) = \frac{1}{\rho^2} (\nabla P) \cdot (\rho \vec{\xi}) - \frac{1}{\rho} \nabla (\vec{\xi} \cdot \nabla P) - \frac{1}{\rho} \nabla (\Gamma \rho \nabla \cdot \vec{\xi})
  \]
  where
  \[
  \frac{d^2 \vec{\xi}}{dt^2} = -\mathcal{L}(\vec{\xi}) = -\omega^2 \vec{\xi}
  \]

- In nonadiabatic case we must include energy equation
  \[
  \frac{d \ln P}{dt} = \Gamma_1 \frac{d \ln \rho}{dt} + (\Gamma_3 - 1) \frac{\rho}{P} \frac{dq}{dt}
  \]

Combining this with previous relations

\[
\frac{d}{dt} \left\{ \int_{V} \frac{1}{2} \rho \left| \frac{d \vec{\xi}}{dt} \right|^2 d\tau + \int_{V} \frac{1}{2} \vec{\xi} \cdot \mathcal{L}(\vec{\xi}) \rho d\tau \right\}
\]

\[
= -\int_{V} (\Gamma_3 - 1) \Gamma \Delta s \left[ \frac{d}{dt} \left( \frac{\Delta \rho}{\rho} \right) \right] \rho d\tau
\]
Stability Continued...

- Identify right side of equation as change in mode energy.
- Average over one period -- use integration by parts to move deriv.
  -- use \( T \frac{d \Delta s}{dt} = \Delta (\varepsilon - \frac{1}{\rho} \nabla \cdot \mathbf{F}) \),
  all of this results in

\[
\langle \frac{d \gamma}{dt} \rangle = \frac{1}{11} \int_0^1 dt \int (I_3 - 1) \frac{d \rho}{\rho} \Delta (\varepsilon - \frac{1}{\rho} \nabla \cdot \mathbf{F}) \rho d \tau
\]

still heating vs. cooling
but...

This is a nonlocal criteria for stability (as opposed to thermal criteria)
Work Integral Integrands

Integrand = \( \frac{2}{3} - 1 \frac{\Delta P \Delta (E - \frac{1}{\rho} \nabla \cdot \vec{F})}{\rho} \)

Accretion Rate = 5 \( \dot{M}_{\text{Edd}} \), Base Flux = 0
2nd, 3rd, and 4th \( \gamma \)-Modes

\[ \text{Integrand} \begin{array}{c}
1 \\
0.5 \\
0 \\
-0.5 \\
-1 \\
-1.5 \\
-2 \\
-3 \\
\end{array} \]

\[ \begin{array}{c}
10^6 \\
10^7 \\
10^8 \\
10^9 \\
10^{10} \\
10^{11} \\
10^{12} \\
\end{array} \]

Column Density (g cm\(^{-2}\))

Heating Not much going on down here!

\( t_{\text{excite}} = 1.3 \text{ sec} \)

\( t_{\text{damp}} = 1.8 \text{ sec} \)

\( t_{\text{damp}} = 0.4 \text{ sec} \)
Work Integral Results vs. Accretion Rate

(Base Flux = 0)

\[ k_{\text{excite}} = \frac{\langle \frac{dU}{dt} \rangle}{\text{Mode energy}} = \frac{\int_{V} \left( (\tau_s^{-1}) \frac{\Delta p}{\rho} \Delta (E - \frac{1}{2} \rho \nabla \cdot \mathbf{v}) \right) \rho d\tau}{\frac{1}{2} \int_{V} \omega^2 \mathbf{z} \cdot \mathbf{z} \rho d\tau} \]

2nd, 3rd, 4th, 5th Modes

damped

Excitation time of \(< 1\text{sec} at \sim 34\text{Hz}\)
Work Integral Results vs. Base Flux

(Accretion Rate = $\dot{m}_{\text{Edd}}$)

Mode instability suppressed at $F_{\text{base}} \approx 0.75 \text{ MeV/nuc} \times \dot{m}_{\text{Edd}}$

$2^{\text{nd}}, 3^{\text{rd}}, 4^{\text{th}}$ G-Modes

$\kappa_{\text{excite}}$ ($s^{-1}$)

Base Flux ($F_{\text{burn}}$)
4U 1820-30

- LMXB with 0.06–0.08 M\(_{\odot}\) He WD companion (Rappaport et al. 1987)
- High luminosity, non-bursting state (Clark et al. 1977, Stella, Kahn & Grindlay '84)

Strohmayer & Brown 2002

**Fig. 1.**—RXTE/ASM light curve of 4U 1820–30 prior to and around the epoch of the superburst. A flux of 1 Crab is approximately 75 ASM units.
Observed Frequency Spectrum (using method of Bildsten, Ushomirsky, Cutler 96)

All frequencies split from $w_0 \approx 25 \text{ Hz}$

($n = 5 \text{ ni Edd}, 2^{nd} C$-Mode)

$w_{obs} = |w - m \Omega|$, $w = w_0 \left( \frac{A}{2} \right)^{1/2}$

C: non-rotating sol'n

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**Graph:**

- $\lambda \rightarrow \ell = 1$
- $\lambda \rightarrow \ell = 2$
- $\Omega$

**Axes:**

- Observed Frequency (Hz)
- Neutron Star Spin (Hz)
Conclusions
- The 2nd G-Mode (shallow water wave) may be interesting
  - Quick excitation time
  - Insensitive to bottom boundary
  - Sensitive to local temperature $f \propto T^{1/2}$
- Rotation important for predicting correct frequencies

Temperature and mode stabilities have different criteria!

Future Work
- Hydrogen/Helium envelopes
- Accumulating envelopes
- Post Burst envelopes