Neutron Stars and the Equation of State
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Masses from Observations
• Radio binary pulsars
• X-ray binaries

Radii from Observations
• Thermal Emission
  \[ R_{\infty} = \frac{R}{\sqrt{1 - 2GM/Rc^2}} \]
  • Pulsars
  • X-ray bursts
  • Low-Mass X-ray Binaries
  • Isolated neutron stars
• Absorption lines in X-ray spectra
• Light curves & pulse fractions
• Quasi-Periodic Oscillations (QPOs)
• Pulsar glitches

Nuclear and High-Energy Experiments
• \( R_n - R_p \) from PREX
• Symmetry Energy from GDP, Radioactive Ion Beams
• Heavy Ion Flows
Compact Star Radii from Thermal Observations

Blackbody

\[ L_{XBB, \infty} = 4\pi R_{XBB, \infty}^2 \sigma T_{XBB, \infty}^4 \]

\[ R_{\infty} = R / \sqrt{1 - 2GM/Rc^2} \]

\[ T_{\infty} = T \sqrt{1 - 2GM/Rc^2} \]

\[ L_{\infty} = L(1 - 2GM/Rc^2) \]

Optical Rayleigh-Jeans Tail

\[ L_{opt, \infty} = 4\pi R_{opt, \infty}^2 \sigma' T_{opt, \infty} \]

\[ L_{optBB, \infty} = 4\pi R_{XBB, \infty}^2 \sigma' T_{XBB, \infty} \]

Optical Excess

\[ f = L_{opt, \infty} / L_{optBB, \infty} \sim 5 - 10 \]

\[ R_{\infty} = \sqrt{R_{XBB, \infty}^2 + R_{opt, \infty}^2} \]

\[ = R_{XBB, \infty} \sqrt{1 + \frac{fT_{XBB, \infty}}{T_{opt, \infty}}} \]

\[ \frac{T_{XBB, \infty}}{T_{opt, \infty}} \sim 2 \]

\[ \frac{R_{\infty}}{R_{XBB, \infty}} \sim 3 - 4 \]
<table>
<thead>
<tr>
<th>Source</th>
<th>Period (s)</th>
<th>$kT_{\text{BB}}$ (eV)</th>
<th>$R_{\infty \text{BB}}$ (km)</th>
<th>Optical Excess</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>RX J1856.5-3754</td>
<td>8.39</td>
<td>61</td>
<td>4.4_{10}^{13}</td>
<td>26.8</td>
<td>Burwitz et al. 2003</td>
</tr>
<tr>
<td>RX J0720.4-3125</td>
<td>5.16</td>
<td>81</td>
<td>6.1_{5}^{13}</td>
<td>26.8</td>
<td>Pavlov et al. 2002</td>
</tr>
<tr>
<td>RX J1308.6+2127</td>
<td>5.16</td>
<td>91</td>
<td>6.5_{5}^{13}</td>
<td>28.7</td>
<td>Kaplan et al. 2002</td>
</tr>
<tr>
<td>RX J1005.3-3249</td>
<td>92</td>
<td>92</td>
<td>3.3_{5}^{13}</td>
<td>27.1</td>
<td>Kaplan et al. 2003</td>
</tr>
</tbody>
</table>
RX J1856.5−3754: Chandra LETGS 502 ks BB fit

counts/sec/keV

energy (keV)

χ

burwitz 12-Nov-2001 15:12
The two-component BB model is schematic only. It is no substitute for realistic atmosphere calculations.

Heavy-element atmospheres more closely resemble BBs than light-element atmospheres, and have reduced emissivities compared to BB. The optical excesses predicted by light-element atmospheres are factors of 100 while those of heavy-element atmospheres are of order 3 to 5.

However, non-magnetic heavy-element atmospheres predict line features apparently not observed (XMM or Chandra). Possible solutions:

- Pavlov et al. (2002) suggest an optically thin admixture of H in a heavy-element atmosphere. However, this must be finely tuned, and the problems of accretion and gravitational settling exist.
- Zane et al. (2003) suggest that cool neutron stars endowed with high magnetic fields suffer phase transitions in their atmospheres, leaving them bare of a gaseous atmosphere. Spectrum is well-fit (including lack of line features) by a bare Fe surface which emits with only \(\sim 30\%\) of BB power.
General Constraints on Structure

- GR: \( R > R_{sh} = 2GM/c^2 = 2.95\frac{M}{M_\odot} \text{ km} \)
- GR + causality: \( R \geq 1.52R_{sh} = 4.48\frac{M}{M_\odot} \text{ km} \)
  (Lattimer et al. 1990; Glendenning 1992)
- GR: \( M_{max} < 4.2\sqrt{\frac{\rho_s}{\rho_o}} \text{ M}_\odot \) (Rhoades & Ruffini '74)
- Binary Pulsar PSR 1913+16: \( M_{max} > 1.442 \text{ M}_\odot \)
- \( R \) independent of \( M \) in range 0.5–1.5 \( M_\odot \)
- Wide range in \( R_{1.4} \): 9–16 km
- No \( M_{max} - R_{1.4} \) correlation: "stiffness"?
- \( d\ln P / d\ln \rho \approx 2 \): polytrope \( n \approx 1 \)
- Wide variation: \( 1 < P(\rho_s)/\text{MeV fm}^{-3} < 6 \)

Newtonian polytropic relations:
\[
P = K \rho^n = K \rho^{1+1/n}
\]
\[
R \propto K^{-n/(3-n)} M^{(1-n)/(3-n)}
\]
With \( n \approx 1 \), \( R \propto K^{1/2} M^0 \propto P^{1/2} \rho^{-1} \).

GR phenomenological result (Lattimer & Prakash '01):
\[
R \propto K^{1/4} \propto P^{1/4} \rho^{-1/2}
\]
Important Exact Analytic Solutions to TOV Equation

1. Incompressible Fluid \((\rho = \rho_c)\)

\[
R = \left( \frac{3GM}{4\pi \rho_c} \right)^{1/3}
\]

\[
\beta \equiv \frac{GM}{RC^2} < \frac{4}{9} \quad \Rightarrow \quad P < \infty
\]

Acausal everywhere, \(\rho_{surface} \neq 0\)

2. Tolman VII (1939) \(\rho = \rho_c[1 - (r/R)^2]\)

\[
R = \left( \frac{15GM}{8\pi \rho_c} \right)^{1/3}
\]

\[
\beta < 0.3862 \quad \Rightarrow \quad \rho < \infty
\]

\[
\beta < 0.2698 \quad \Rightarrow \quad c_s^2 < c^2
\]

3. Buchdahl (1967) \(\rho c^2 = 12\sqrt{PP_* - 5P}\)

\[
R = (1 - \beta) \sqrt{\frac{\pi c^4}{288P_*G (1 - 2\beta)}}
\]

\[
0 < \beta < 2/5 \quad \Rightarrow \quad \rho > 0
\]

\[
\beta < 1/6 \quad \Rightarrow \quad c_s^2 < c^2
\]

4. Nariai IV (1950)

\[
\beta < 0.4126 \quad \rho < \infty
\]

\[
\beta < 0.2277 \quad c_s^2 < c^2
\]
Pressure of beta-stable matter

\[ E(n,x) \simeq E \left( n, \frac{1}{2} \right) + S_v(n) (1-2x)^2 + \ldots \]

\[ P(n,x) = n_s \left( \frac{\partial E(n,x)}{\partial n} \right)_x = n^2 \left[ E' \left( n, \frac{1}{2} \right) + S'_v(n) (1-2x)^2 + \ldots \right] \]

Leptons: \( E_e = \frac{3}{4} \hbar c x (3\pi^2 n x)^{1/3} \)

Beta equilibrium: \( \mu_e = \mu_n - \mu_p = -\frac{\partial E}{\partial x} \)

\[ x \simeq \left( 3\pi^2 n \right)^{-1} \left( \frac{4S_v}{\hbar c} \right)^3 \quad \text{At } n_s, x_s \simeq 0.04 \]

\[ P(n_s) = n_s (1 - 2x_s) \left[ n_s S'_v(n_s) (1 - 2x_s) + x_s S_v(n_s) \right] \]

At higher densities:

\[ E \left( n, \frac{1}{2} \right) \simeq -16 + \frac{K}{18} \left( \frac{n_s}{n} - 1 \right)^2 + \frac{S}{162} \left( \frac{n_s}{n} - 1 \right)^3 + \ldots \]

\[ P \left( \frac{3}{2} n_s \right) \simeq \frac{9}{4} n_s \frac{K}{18} \left[ 1 + \frac{S}{12K} + n_s (1-2x)^2 S'_v \left( \frac{3}{2} n_s \right) + \frac{2}{3} x S_v \left( \frac{3}{2} n_s \right) (1-2x) + \ldots \right] \]

\( K \approx 200 - 250 \text{ MeV} \)

\( -S \approx 1500 - 2500 \text{ MeV} \)
"Standard" Neutron Star Cooling vs. Present Data

Each shaded area represents 45 different pairing models:
3 different $n^1S_0$ gaps x 3 different $n^3P_2$ gaps x 5 different $p^1S_0$ gaps

EOS: APR $M = 1.4 M_\odot$
Maximum Rotation Rate

Non-relativistic Roche Model (Shapiro & Teukolsky)

\[ \Omega_K^2 = \frac{G M}{R_K^3} \]

Keplerian shedding limit

\[ R_K = \frac{3}{2} R_0 \]

\[ V_K = \frac{1}{2\pi} \left( \frac{2}{3} \right)^{3/2} \left( \frac{G M}{R_0^3} \right)^{1/2} \]

\[ = 9.98 \left( \frac{M}{M_0} \right)^{1/2} \left( \frac{10 \text{km}}{R_0} \right)^{3/2} \]

GR result

Maximum spin rate for a given EOS (maximum mass enhanced)

\[ V_{K,\text{max}} \approx 1225 \pm 40 \left( \frac{M_{\text{max}}}{M_0} \right)^{1/2} \left( \frac{10 \text{km}}{R_{\text{max}}/2} \right)^{3/2} \text{Hz} \]

(Haensel & Zdunik 1990)

(Lattimer et al. 1990)

Maximum spin rate for a given mass (nearly independent of EOS if far from maximum mass)

\[ V_K \approx 1045 \pm 30 \left( \frac{M}{M_0} \right)^{1/2} \left( \frac{10 \text{km}}{R_0} \right)^{3/2} \text{Hz} \]

\( R_0 < 15.5 \left( \frac{640}{V_K} \right)^{1/3} \left( \frac{M}{M_0} \right)^{1/3} \text{km} \)

\( R_0 < 17.6 \left( \frac{750}{V_K} \right)^{1/2} \left( \frac{M}{2M_0} \right)^{1/2} \text{km} \)
Maximum Possible Density in Compact Objects

If EOS "known" up to fiducial density $\rho_0$ and "causal" above $\rho_0$

$$M_{\text{max}} = 4.1 \left( \frac{\rho_5}{\rho_0} \right)^{\frac{1}{2}} M_\odot$$

(Rhoades & Ruffini 1974)

$$\rho_5 = 2.7 \cdot 10^{15} \text{ g cm}^{-3} \text{ nuclear saturation density}$$

$$R_{\text{max}} \geq \frac{3G M_{\text{max}}}{c^2} = 18.1 \left( \frac{\rho_5}{\rho_0} \right)^{\frac{1}{2}} \text{ km}$$

(Hultsman et al. 1990)

Most "compact" EOS is incompressible fluid. Couple with "causal" constraint:

$$\rho_{c, \text{inr}} = \frac{3}{4\pi} \left( \frac{c^2}{3G} \right) \frac{1}{m^2} = 5.5 \cdot 10^{15} \left( \frac{M_\odot}{m} \right)^2 \text{ g cm}^{-3}$$

This EOS violates causality

Tolman VII: $P = \rho_c \left[ 1 - \left( \frac{c^2}{2G} \right) \right]$

$$\rho_{c, \text{II}} = \frac{15}{8\pi} \left( \frac{c^2}{3G} \right) \frac{1}{m^2} = 13.8 \cdot 10^{15} \left( \frac{M_\odot}{m} \right)^2 \text{ g cm}^{-3}$$

No "realistic" EOS has a greater $\rho_c$.

Implies $\rho_{c, \text{II}}$ is the maximum density possible.

$$2.2 M_\odot \Rightarrow P_{\text{max}} = 2.9 \cdot 10^{15} \text{ g cm}^{-2}$$
Conclusions

Radius is a powerful diagnostic of EOS in the vicinity of $P_{\text{sat}}$.

Direct determination of pressure
$\Rightarrow$ density dependence of symmetry energy

Possible appearance of exotic phases

Mass is a constraint on strange quark matter.

Maximum mass $\Rightarrow$ maximum possible density at zero temperature

Temperature and age can constrain existence of exotic phases + superfluidity.